The Centre For Conceptual Sculpture

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Among my morning mail was a letter in a triangular envelope. I doubt that the post office approved of this unusual shape — it probably didn't fit very well in the automatic sorting machines — but it certainly attracted my attention. So did the letterhead. The logo was a hole in the shape of an annulus — a thick ring. I spent some time trying to work out how they kept the circular non-hole in the middle of the annulus in place, but gave up, baffled. The letter was from an outfit that called itself the Centre for Conceptual Sculpture. In rather convoluted language it told me that the Centre, still under construction, had run into some teething troubles. Would I help them sort everything out?

I sent a quick fax to say 'yes' and headed for the station.

The Director of the Centre was a gentle giant of a man, well over six feet tall with a huge orange beard, by the name of Scrimshaw Whittler. He started to explain some esoteric problem that was holding up the building work, but I couldn't understand a word, so I stopped him. "I think you'd better start at the beginning, Mr. Whittler. What precisely *is* conceptual sculpture?"

"Ah, yes. Perhaps I had better show you round first, so that you get an idea of what the purpose of this building is." He led me down some connecting corridors, past a door bearing a large numeral '1'. "Not that one," he muttered. "Too trivial. I think you will gain a better understanding from rooms 2 and 3."

We entered room 2.

It was crammed full of large white boards, ruled into squares with thin black lines, upon which were painted large coloured dots. They were lying on the floor, nailed to the walls, glued to the ceiling, and propped up against each other in piles like a house of cards. I took a close look at the nearest one. There were four red dots, at the corners of one of the ruled squares. The caption read "*Square 1*, Alexander Tripe, 1973." Next to it was another with four dots, in a square twice the size: "*Square 2*, Alexander Tripe, 1973." I walked along until I reached "*Square 22*, Alexander Tripe, 1973". Next to it was a large area of blank wall.

"I see you are attracted by our Tripe collection," said Scrimshaw Whittler. "It is the largest in Europe. Unfortunately the price doubles for each piece, and our funds were insufficient to purchase '*Square 23*'. It went to a Japanese electronics magnate."

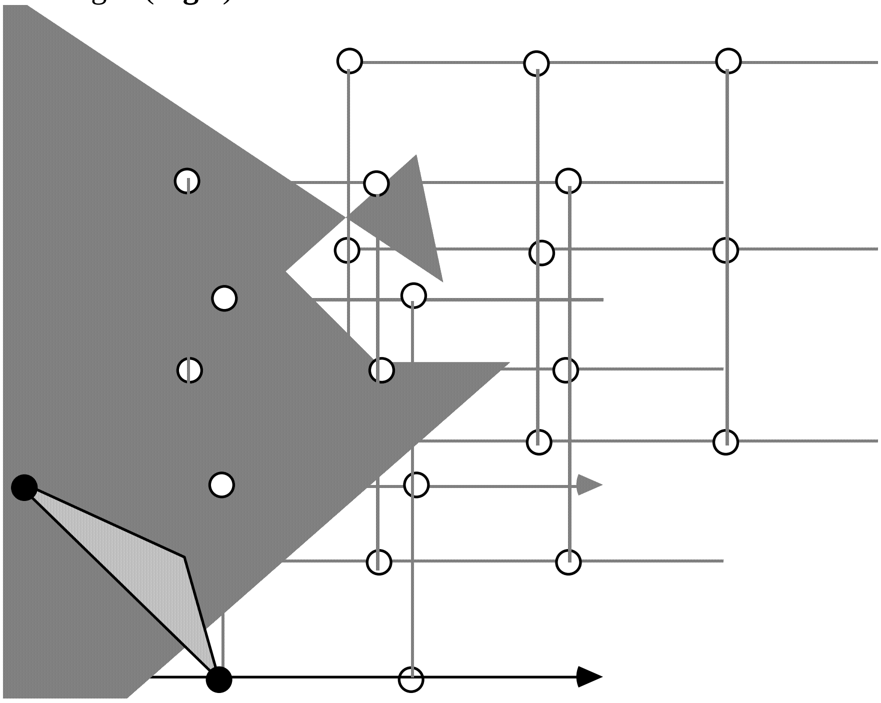
"Sad," I said. "Do you have anything other than Tripe?"

"Oh, yes! We have a great deal of of Bilge and a fair amount of Shenanigans. I particularly like this one, don't you?" The caption read "*Nearly Equilateral Triangle*, Seamus Shenanigan, 1988." The item consisted of three red dots, arranged at roughly equal distances apart at intersections of thin black grid lines.

"What happened to his '*Equilateral Triangle*'? I asked, thinking I was making a joke."

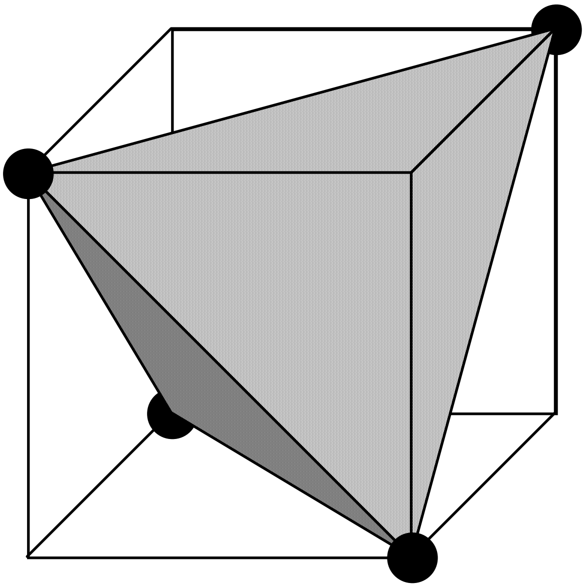
"Follow me," said Scrimshaw Whittler. He led the way into room 3.

Like room 2 it was littered with grids; but instead of being drawn on white boards with black paint, these were made from metal rods painted black. The rods were arranged in a cubic lattice, like the edges of a three dimensional chessboard. Attached to them, at intersections of grid-lines, were red balls. We soon found Shenanigan's 'Equilateral Triangle' (**Fig.1**).



Shenanigan's '*Equilateral Triangle*'.

"It has to go in room 3," the Director explained. "Édouard Lucas proved in 1878 that it is not possible to have an *equilateral* triangle in room 2. Which is really rather strange, when you consider that it *is* possible to have a tetrahedron in room 3." He led the way to '*Tetrahedron*, Thomas Rot, 1985' (**Fig.2**).



Thomas Rot's '*Tetrahedron*'.

"We have a lot of Tommy Rot too," he explained proudly.

It was beginning to make sense, in a strange sort of way. "Show me room 4," I said. I wasn't sure what I was expecting — maybe computer screens with revolving shapes, something along those lines — but it wasn't what lay beyond the door marked '4'.

Room 4 looked like a paper-recycling factory. Sheets of paper were piled everywhere. "It gets *very* conceptual from room 4 onwards," the Director said apologetically. I picked up one sheet. It was covered in numbers, arranged in fours, like (1,0,-1,2)
or more ambitiously

(243, -9975, 42, 100000001).

"Lattice polyhedra!" I yelled. " A conceptual sculture is just a lattice polyhedron! And the number of the room is the dimension of the lattice!"

He nodded. "You see *why* we have to be very conceptual in rooms 4 onwards."

"Of course. An n-dimensional lattice can be thought of as the set of all n-tuples of whole numbers, positive or negative. Integers. Lattice polyhedra have their vertices at points of the lattice. Only in two and three dimensions — your rooms 2 and 3 — can you actually make physical models."

"In room 1 also," he said. "But it's not a very interesting room, unless you like minimalist sculpture. Conceptual sculpture was the brainchild of Alexander Tripe, back in the late 1960s. He wanted to make four-dimensional sculptures, but found it hard to obtain the necessary materials. For a time he produced sequences of three-dimensional sculptures, changing systematically from one to the next, rather like representing a movie as a sequence of static frames. But he wasn't satisfied. Then he hit on the idea of representing the sculpture in the abstract as a set of coordinates. Because of the inherent inaccuracy of decimal-place numbers, he restricted the coordinates to be integers. Here is the pinnacle of our collection, his very first full-fledged four-dimensional conceptual sculpture."

A square of cork was attached to the wall, inside a large glass case. No less than five security cameras were trained on it. Inside was an instrument to monitor temperature and humidity. A piece of paper torn roughly from the edge of a magazine was pinned to the cork. On it was written, in scruffy handwriting,

(1,0,0,0)
(0,1,0,0)
(0,0,1,0)
(0,0,0,1).

The caption read '*Tetrahedron in 4-space*, Alexander Tripe, 1967'. "It is a masterpiece," sighed Scrimshaw Whittler.

"I think I've got the idea," I said. "Now you can explain your problem, and I'll do my best to answer it."

"In my office," he said. "It will be easier to talk there."

The office was attached to a penthouse apartment, beautifully and expensively furnished. He noticed me gazing at the gold-plated door-handles. "Great art deserves only the best," he said. "Therefore so do its custodians."

We sat down in deep, soft leather armchairs. "Right," I said. "Tell me the problem."

"There are many," he sighed. "For example, *this*." He tossed something flat and hard at me. I caught it by reflex. It was a pentagon.

"So?" I asked.

"Which *room* does it go in?"

"It's a *regular* pentagon, yes? All sides perfectly equal, all angles perfectly equal?"

"You got it."

"You want to know what dimension of lattice contains five points that form a regular pentagon."

"Precisely. We know that you can get a square in room 2, and equilateral triangles and regular hexagons in room 3 — but there our knowledge ceases." [*Find how to fit a hexagon in room 3.*]

"Crystallographic restriction," I said.

"Pardon?"

"The atoms of crystals form regular lattices. Rotational symmetries of crystals correspond to regular polygons in those lattices. The crystallographer William Barlow proved over a century ago that no crystal has an axis of fivefold symmetry — or indeed, *any* n-fold rotational symmetry other than n = 1, 2,3,4 and 6. His proof was aimed at lattices in three dimensions; but very similar methods work in any number of dimensions. In 1937 Isaac Schoenberg proved that the only regular polygons embeddable in a lattice, of any dimension, are those with 1, 2, 3, 4, and 6 sides. The cases n = 1 and 2 of course are not conventionally associated with polygons: here a 1-gon is a point and a 2-gon is a line interval."

"You are telling me that it can't be done?"

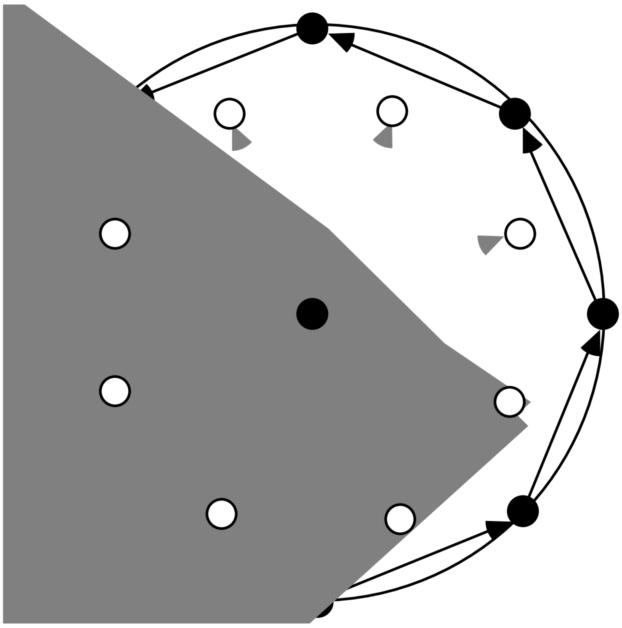
"That's right."

"But that is terrible! Conceptual sculptures do not include pentagons? Where am I to get my military sponsorship from? The Americans will be disgusted!"

"I'm sorry, but that's how it goes."

"I will need to be convinced," he said, decisively.

"There's a beautful proof, due to W.Scherrer in 1946. Assume that there exists a regular n-gon in a lattice. First, suppose n > 6. Consider the lattice vectors fromed by the n sides of the polygon. Translate them so that their tails are all at the origin. Then their heads form a *smaller* lattice n-gon (**Fig.3a**).

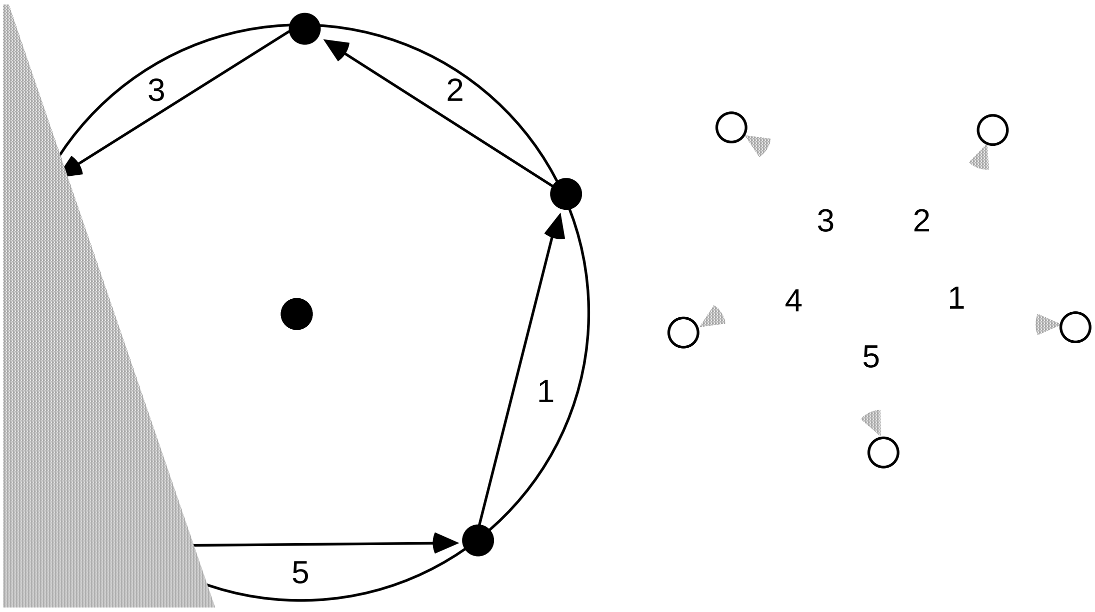


(a) Proof that no lattice contains a regular n-gon, n _ 7. Transport the sides (black arrows) to the centre (grey arrows) to create a smaller lattice polygon (white dots).

Repeating this idefinitely, we obtain an infinite decreasing sequence of lattice n-gons. But this is absurd, because there is a definite minimum distance between lattice points."

"You have missed out the pentagon."

"When n = 5 you just have to be slightly more clever. Number the sides of the pentagon in order, 1, 2, 3, 4, 5. Taking them in the order 1, 3, 5, 2, 4 place the tail of eahc at the head of the previous one. This produces a five-pointed star, whose corners from a lattice pentagon smaller than the original (**Fig.3b**).



(b) Proof that no lattice contains a regular pentagon. Transport the sides (black arrows) to form a star (grey arrows) whose vertices (white dots) form a smaller lattice polygon.

The same contradiction follows."

The Director of the Centre for Conceptual Sculpture hung his head in his hands.

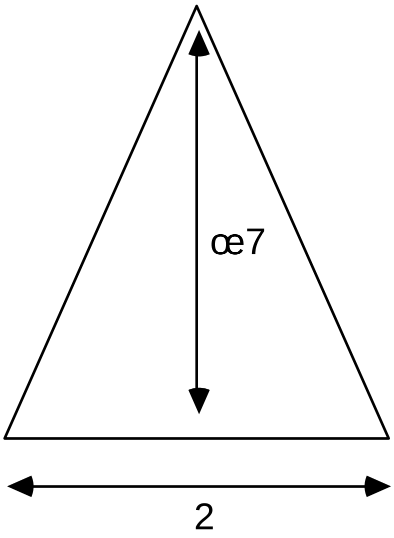
"Never mind," I said. "The star of David is a lattice polygon." [*How?*] "You can get funding from Israeli Intelligence instead." A glimmer of a smile returned to his face.

"Actually," I continued, "Schoenberg's 1937 paper wasn't really about polygons, but about simplexes. A 2-dimensional simplex is an equilateral triangle; a 3-dimensional simplex is a tetrahedron; and in general an n-dimensional simplex consists of n+1 points in n-space, all equal distances apart. Every n-simplex easily embeds in an (n+1)-dimensional lattice — much as for Tripe's '*Tetrahedron in 4-space*'. Schoenberg asked when an n-simplex can be embedded in an n-dimensional lattice — one dimension lower. The equilateral triangle can't be embedded in a planar lattice — I'll explain why in a moment — but the tetrahedron *can* be embedded in a 3-dimensional lattice, as Rot's '*Tetrahedron*' shows."

"Schoenberg showed that the answer is extremely curious. For example, if n _ 25 then the n-simplex embeds in an n-dimensional lattice if and only if n = 1, 3, 7, 8, 9, 11, 15, 17, 19, 23, 24, 25." [*Can you spot any patterns?*]

Scrimshaw Whittler shook his head in bewilderment. "I can see that which room we put a given polyhedron in is quite a subtle business," he said. "I had rather hoped that it would be simpler than that." He rummaged around the room and found a large cardboard box. "So what do I do with these?" He pulled out — a triangle. The box was full of triangles of every conceivable shape. "They're supposed to go in a new wing we're currently seeking donations for, the Triangle Galleries. But *where*?"

He handed one to me (**Fig.4**).



Which dimension of lattice contains a triangle similar to this?

On it was written "isosceles triangle of height œ7 and base 2."

"You want to find three lattice points that form a triangle of this size and shape?"

"Shape. The size is immaterial."

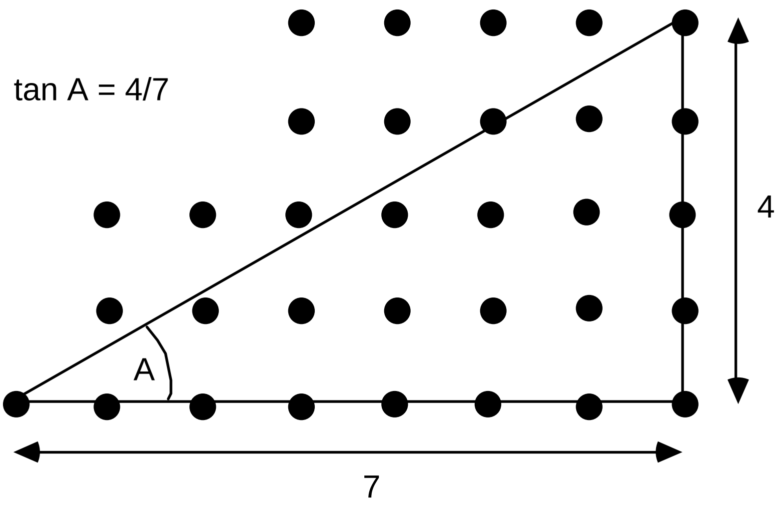
"Ah. You want a triangle similar to this one, with all vertices at lattice points. And you want to know what dimension of lattice will do the trick."

"Precisely."

I thought about that. "Suppose we define an *n-lattice triangle* to be one that is similar to a triangle with all vertices in some n-dimensional lattice. What you need is a characterisation of the n-lattice triangles, for each n. Hmmmm. Well, I can see that every n-lattice triangle is also an m-lattice triangle whenever m _ n, so it's the smallest possible n that really counts. And because we're talking similar triangles here, the only thing that matters is the *angles*. Let's think about lattices in the plane — 2-lattice triangles. Oh, I see, it's the *tangent* of the angle that matters."

Scrimshaw Whittler looked perplexed. "I knew curves had tangents, but angles?'

"Trigonometry. The tangent of an angle is the ratio of the opposite and adjacent sides when the angle is drawn in a right triangle (**Fig.5a**).

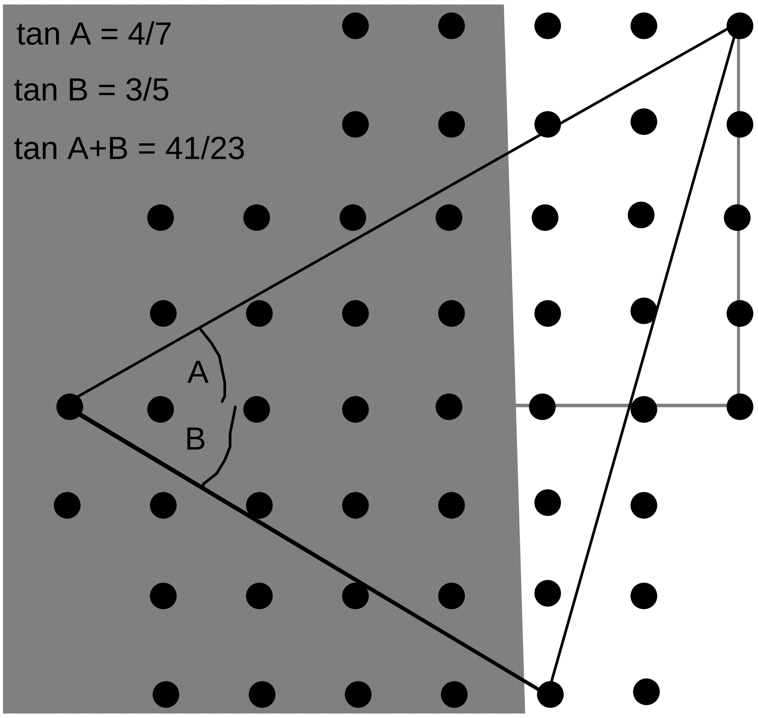


(a) Angle A has rational tangent.

If the vertices of the right triangle are lattice points, then the tangent must be a rational number — a ratio of whole numbers. Because the sides of the right triangle are whole numbers."

"Right."

"Of course, the lattice triangle may be tilted. But then each angle can be split into two that do have rational tangents (**Fig.5b**).



(b) So does angle B, hence also angle A+B since tan(A+B) = $\left(\frac{4}{7} + \frac{3}{5}\right) / \left(1 - \frac{4}{7} \text{¾} \frac{3}{5}\right) = \frac{41}{23}$.

And there's a formula for the tangent of a sum of two angles,

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} .$$

That formula imples that if tan A and tan B are rational, then so is tan(A+B). So every angle of a 2-lattice triangle has a rational tangent.

"In fact, John McCarthy recently showed that the converse is true as well. The proof is to drop a perpendicular to produce two right triangles, and then run the tangent argument backwards. In short, 2-lattice triangles are precisely those whose angles have rational tangents."

Scrimshaw Whittler sighed. "That helps?"

"Of course. For example, the equilateral triangle has angles of 60_. Since tan 60_ = œ3, which is irrational, the equilateral triangle is not a 2-lattice triangle. On the other hand, it *is* a 3-lattice triangle, as Shenanigan discovered. And *this* triangle—" I waved the one that he'd pulled from the box, the isosceles triangle of base 2 and height œ7 — "has base angles with tangent œ7, also irrational, so it's not a 2-lattice triangle either!"

Scrimshaw Whittler offered a weak smile. "I really need to know more than that. I need to know exactly which dimension of lattice works for any given triangle."

"I know. Wait, it's all coming back to me now. Michael Beeson at San Jose State University solved the whole problem only a few months ago. He got very strange answers. Even though there are infinitely many different dimensions n for the lattice, there are only three distinct cases: 2 dimensions, 3 or 4 dimensions, and 5 or more dimensions.

- (n=2) A 2-lattice triangle is precisely one for which the tangents of its angles are rational.
- (n_5) A 5-lattice triangle is precsiely one for which the *squares* of the tangents of its angles are rational, and the same is true for all larger n.
- (n=3 or 4) A triangle is a 3-lattice triangle if and only if the tangents of its angles are all rational multiples of œk, where k is a sum of three integer squares. The same condition holds for 4-lattice triangles.

Beeson found his results through computer experiments, but then managed to prove everything using number theory. For example, the fact that all lattice triangles can be embedded in a 5-dimensional lattice is a consequence of the theorem that every whole number is a sum of four perfect squares. The number k that appears when n = 3 or 4 is related to the area of the triangle."

Scrimshaw Whittler looked baffled. "Say that again."

"It all depends on the tangents of the angles, and there are three cases. If a triangle embeds in an n-dimensional lattice, for any n, then it must embed in a 5-dimensional lattice. There are some 5-lattice triangles that are not 4-lattice triangles; but any 4-lattice triangle must also be a 3-lattice triangle. Finally, there are some 3-lattice triangles that are not 2-lattice triangles. And by calculating the tangents of the angles and using number theory, you can decide exactly which is which.

"For instance, your isosceles triangle has angles whose tangents are all rational multiples of œ7. So the squares of the tangents are rational, and it must be a 5-lattice triangle. However, since 7 is *not* a sum of three squares, your isosceles triangle is not a 4-lattice triangle. It belongs in room 5, Director." [*Can you find three points in the 5–dimensional lattice that form such a triangle?*]

Scrimshaw Whittler leaped to his feet and shook my hand warmly. "I must congratulate you! My problems are at an end, and construction of the Triangle Galleries can stop with room 5!"

"Don't congratulate *me*," I said. "Michael Beeson did all the work. You do realise that as far as triangles go, rooms 3 and 4 are the same? You could use two doors, each with one of the two numbers."

"Yes! That will save millions!" He glanced down at his cardboard box of triangles. "I can find a room in the Triangle Galleries for every single one!"

"Um — Director?"

"Yes?"

"You did say every *conceivable* triangle is in there?"

"Of course. Here's one given to us by Archimedes, used in his proof of the area of a circle. It is a right triangle of height 1 and base 2_. Its will be the pride and joy of the Triangle Galleries!"

I coughed in embarrassment. "I'm sorry, but it won't."

"Why ever not?"

"It has an angle whose tangent, squared, is $4\_^2$. That is *irrational*. Archimedes' triangle is not an n-lattice triangle for any n. Beeson's theorem says that *if* a triangle is a lattice triangle, *then* it is a 5-lattice triangle. But it doesn't say that *all* triangles are n-lattice triangles."

For a second, Director Whittler looked wistfully at Archimedes' triangle. Then... "Oh well," he said, tossing it into the wastebasket. "Nobody will miss one little triangle."

"Yes, but that's not the only one," I said. "Rather a lot of triangles have angles whose tangents, squared, are irrational."

"How many?" he said, looking worried.

"Almost all of them," I said.

He stared at me. "You know what *I* think?" he asked.

"No. What?"

"*I* think we need to hire a different consultant." He rose menacingly to his feet. The giant no longer looked quite so gentle. The orange beard bristled with fury.

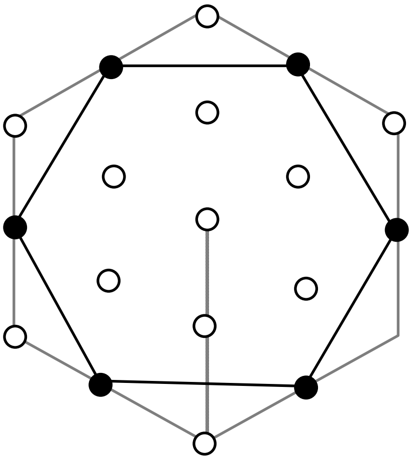
I backed away towards the door. :"It won't alter the truth," I said.

"No," he replied. "But it *will* alter how much of it becomes public knowledge."

I made a strategic withdrawal at high speed.

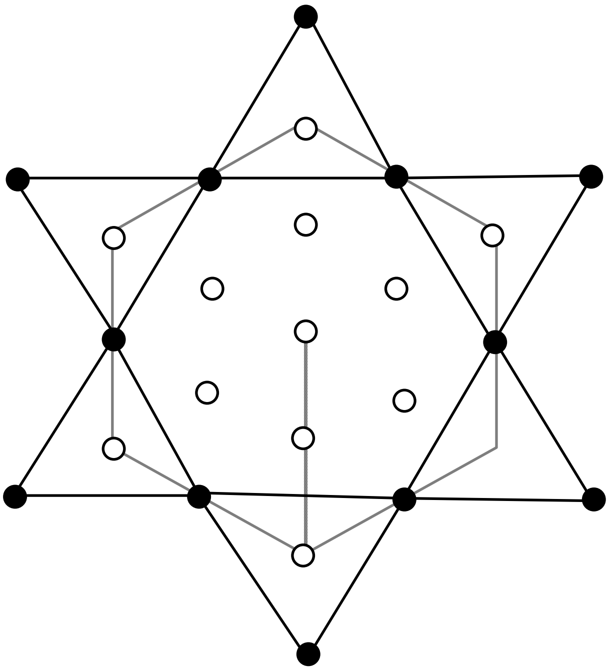
ANSWERS

1 A 3-lattice hexagon is shown in **Fig.6**.



How to find a hexagon in a 3-dimensional lattice. Select suitable midpoints of a 2_2 lattice cube.

2 A 3-lattice star of David is shown in **Fig.7.**



A star of David in a 3-dimensional lattice. Just extend the sides of the hexagon in Fig.5 to meet in pairs.

3 Schoenberg proved that an n-simplex embeds in a n-dimensional lattice if and only if:

- n even: n+1 is a perfect square
- n of the form 4m+3: any n

- n of the form 4m+1: n+1 is a sum of two perfect squares.

4 It is necessary to take n _ 5. When n = 5 the points (1,0,0,0,0), (-1,0,0,0,0), and (0,1,1,1,2) form such a triangle.

FURTHER READING

Michael J. Beeson, Triangles with vertices on lattice points, *American Mathematical Monthly* **99** (1992) 243-252.

E.Lucas, Théorème sur la géometrie des quinconces, *Bulletin de la Societé Mathématique de France* **6** (1878) 9-10.

W.Scherrer, Die Einlagerung eines regulären Vielecks in ein Gitter, *Elemente der Mathematik* **1** (1946) 97-98.

I.J.Schoenberg, Regular simplices and quadratic forms, *Journal of the London Mathematical Society* **12** (1937) 48-55.