Divide (or Multiply) And Conquer
or
Juniper Green

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Elegance and simplicity go hand in hand. About a year ago I was talking to Ian Porteous, a mathematician at the University of Liverpool in the UK, and he told me about one of the simplest and most elegant mathematical games that I have ever encountered. It was invented about ten years ago by his son Richard Porteous, and is called 'Juniper Green' after the school at which Richard was a teacher. He invented the game in order to teach young children about multiplication and division using small whole numbers. The rules are natural and simple, and the game is fun to play. Even adults find it more subtle than they expect, and recreational mathematicians will find the search for a winning strategy to be surprisingly challenging. Moreover, there are some interesting unsolved problems, and I confidently expect a large mailbag for 'Feedback'.

To play Juniper Green, you should make 100 cards, numbered 1 through 100. Lay them face up on the table in numerical order, say ten rows of ten numbered 1-100 from left to right and top to bottom like words on a printed page. The precise layout is unimportant, but it should be easy for players to locate the desired card.

Here are the rules.

1. Two players take turns to remove one card from the table. Cards removed are not replaced and cannot be used again.
2. Apart from the opening move, each number chosen must either be an exact divisor of the previous player's choice, or an exact multiple.
3. The first player who is unable to choose a card loses.

There is one final rule, which I will supply in a moment. Without it, the game could easily be won by the first player using what I shall call the double whammy tactic. It depends upon using primes. Recall that a prime number is one with no divisors other than itself and one. The double whammy relies upon 'big primes', by which I mean primes larger than 50 (half of 100). If a player picks a big prime, then the next player is forced to pick card 1. Suppose Alice plays against Bob, with Alice going first. She plays a big prime — say 97. Bob must play 1. Now Alice plays another big prime — say 89. At this point Bob has used up card 1, and is stuck. (Note that this idea fails with smaller primes — for example 47 has the reply 94 as well as 1.)

To prevent this spoiling strategy, there is a fourth rule:

4. The opening move in the game must be an even number.

(Subsequent moves may either be even or odd.)

With this addition to the rules, the game becomes not just playable, but interesting — at least until one of the players figures out a winning strategy. Even though the game starts with an even number, at later stages odd numbers can occur, so big primes still influence play. In particular, if any player picks card 1 then they lose, assuming their opponent is awake. However, they lose by a single whammy, not a double one. If Bob chooses 1 (which is always available to him because it divides everything else) then Alice responds with a big prime, say 97, and Bob has nowhere to go. At least, that's true provided we answer one question. How do we know that 97 has not been taken already? Well, it can only be chosen if the previous player chooses 1: it neither divides, nor is a multiple of, anything else except 97 itself. But since Bob has just chosen 1, no previous player can have done so, and we know 97 is still sitting there waiting for Alice to pounce. In short, players should always avoid picking 1 if they can, and with sensible play thegame ends when a player is forced to choose card 1.

BOX 1 shows a sample game, played to illustrate the rule \and without much regard for good tactics.

BOX 1 ================================

Move #	**Alice**	**Bob**	**Comments**
1	48		Even number, as required by rule 4
2		96	Doubles Alice's choice
3	32		One third of Bob's choice
4		64	Bob is forced to choose a power of 2
5	16		So is Alice
6		80	Multiply by 5
7	10		Divide by 8
8		70	Multiply by 7
9	35		Halve
10		5	Only choices are 7, 5 (or 1 and lose)
11	25		
12		75	Only 50 and 75 available
13	3		
14		81	
15	9		Only 27 and 9 available
16		27	Bad move!
17	54		Forced since 1 loses
18		2	Forced
19	62		Inspired variant on big prime whammy
20		31	Forced
21	93		Only choice but a good one
22		1	Forced, and loses, because...
23	97		Big prime whammy

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My advice to you now is to stop reading, make a set of cards, and play the game for a while. Although I'm not going to give away the winning strategy — I'll put it in a subsequent Feedback section so as not to spoil your fun — I *am* going to analyse the same game when there are only 40 cards numbered 1-40, and the analysis will give you some broad hints on the 100-card game too.

One of the attractive features of Juniper Green is that it can be played with any number of cards. Very young children might use a pack numbered 1-20, Juniper Green experts would go for at least a 1000-card deck. Who should win, with the best tactics, and how they must play to do so, depend on the number of cards. For brevity, let me call n-card Juniper Green 'JG-n', so that the standard game just described is JG-100. For illustrative purposes, let's try to find a winning strategy for JG-40.

First, it is clear that some opening moves lose rapidly. For example:

Move #	**Alice**	**Bob**
1	38	
2		19
3	1	
4		37
5	LOSES	

The same goes for an opening move of 34.

Moreover, some numbers are best avoided — just as 1 is. For instance, suppose that Alice is unwise enough to play 5. Then Bob strikes back with a vengeance:

Move #	**Alice**	**Bob**	
?	5		
?+1		25	
?+2	1		Forced and losing.

Note that 25 *must* still be available,because it can only be chosen if the previous player plays 1 or 5.

Since Alice knows that she is in trouble if she plays card 5, her obvious tactic is to try to force Bob to play 5 instead. Can she do this? Well, if Bob plays 7 then she can play 35, and Bob has to play 1 or 5, both of which lose. Fine, but can she force Bob to play 7? Yes: if Bob has chosen 3 then Alice can play 21 and that forces a reply of 7. Fine, but how does she make Bob play 3? Well, if he plays 13 then Alice plays 39. Alice can go on in this manner, building hypothetical sequences of moves all of which force Bob's reply at every stage and which lead to his inevitable defeat — but can she maneuver Bob into such a sequence to begin with? At some stage the moves have to involve even numbers, so the card numbered 2 is likely to play a pivotal role. Indeed if Bob plays 2 then Alice can play 26, forcing Bob into the trap of playing 13. So now we come to the crunch. How can Alice force Bob to play 2?

What if Alice opens with 22? Then Bob either plays 2 and gets trapped in the long sequence of forced moves outlined (in reverse order) above; or he plays 11. Now Alice has the choice of playing 1 and losing, or going to 33. When she picks 33, 11 has already been used up, so Bob is forced to 3... and we already know how Alice can win when he does that. So Alice must win if she starts with 22.

BOX 2 summarizes Alice's strategy: the two sets of columns deal with the two alternatives Bob can pick. Assume throughout that all players avoid 1 since it is an instant loss: with this eliminated, virtually every move is forced.

BOX 2 ====================================

Move #	**Alice**	**Bob**	**Alice**	**Bob**	
1	22				
2		11		2	(Bob has two choices)
3	33		26		
4		3		13	
5	21		39		
6		7		3	
7	35		21		
8		5		7	
9	25		35		
10		LOSE		5	
11			25		
12				LOSE	

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There is at least one other possible opening move for Alice that also lets her force a win: if she chooses 26 instead, then the same kind of game develops, but with a few moves interchanged as in BOX 3.

BOX 3 ======================================

Move #	Alice	Bob	Alice	Bob	
1	26				
2		13		2	(Bob has two choices)
3	39		22		
4		3		11	
5	21		33		
6		7		3	
7	35		21		
8		5		7	
9	25		35		
10		LOSE		5	
11			25		
12				LOSE	

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Why does this strategy work? What is it about the numbers 22 and 26 that allows Alice to play them off aginst each other? The crucial features here are the primes 11 and 13. The opening move is twice such a prime — 22 or 26. It forces Bob to reply either with 2 — at which point Alice is away — or the prime. But then Alice replies with three times the prime, forcing Bob to go to 3 — and she's away again. So Alice escapes trouble because as well as twice the prime, there is exactly one other multiple of such a prime that is under 40 — namely 33 or 39. This provides an 'escape route'. Let us call these the medium primes — they lie between one third and one quarter of the number of cards. What Alice discovers is that when she plays twice a medium prime, the Bob must play that prime, and then she can play three times that prime, forcing Bob to play the number 2.

Can Alice win by any other strategy? Does any opening choice different from 22 or 26 also lead to a win? That's for you to find out. Moreover, you are now in a good position to analyse JG-100 — or for the ambitious. JG-1000. Is there a first-player strategy to force a win?

Finally, the time has come to open up the problem in its full generality. Consider JG-n for any whole number n. Because it is a finite-state game, in which no draws are allowed, general game theory implies that either Alice has a winning strategy, or Bob does, but not both. So which is it? Say that the limit n is primary if Alice has a winning strategy at JG-n, and secondary if Bob has. (Remember: Alice goes first.) Can you characterise which n are primary and which are secondary? Certainly the answer depends on n. For very small n, a few quick calculations indicate that 1, 3, 8, and 9 are primary whereas 2, 4, 5, 6, 7are secondary. What about n = 100? What about all the values of n from 10 to 99? And what about completely general n? Can anyone find any patterns, even if they can't prove they are correct? Can anyone solve the whole thing?