

Election Fever in Blockvotia

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The National Assembly of Blockvotia had just counted the votes on the Palmgreasing Slushfund Bill, and President Freebie Perks was not pleased. His aides were doing their best to calm him down, while trying even harder not to offend him and risk their jobs.

"Penny, I don't like to lose a vote we ought to have won."

"Of course not, sir," said his private secretary, Penelope Poundpincher.

"You told me that four out of the six districts were in favour, including Sheepshire, which is the biggest."

"I did, sir. And it was true."

"Then how the devil did we *lose*?"

"It's the weighted voting system that did it," said Penny. "As you know, each district has assigned to it a number of votes that is roughly proportional to its population. Sheepshire has 300,000 people, and gets 10 votes. The other districts — well, here's a print-out that shows all the details (**TABLE 1**). The total vote is 31, so any coalition that together can sum up 16 votes can decide the outcome."

TABLE 1 Voting weights for the districts of Blockvotia

District	number of votes
Sheepshire	10
Richfolk	9
Candlewick	7
Fiddlesex	3
Slurrey	1
Porkney Isles	1

"Sheepshire voted for the bill. So did Fiddlesex, Slurrey, and the Porkneys. Four districts out of the six, as I said, including the biggest. But their total vote is only 15. Richfolk and Candlewick voted against; and their votes total 16. That's what lost us the bill."

President Perks scrutinised the numbers carefully. "Penny, the Presidential Election is coming up next month. The districts form the electoral college, and I don't

want a repeat. If we got the Boundary Commission to give Sheepshire one more vote and Candlewick one less... "

Penny shook her head. "I wouldn't recommend that, sir," she said. "Richfolk and Candlewick both favour your re-election. Sheepshire is wavering, and the other three districts are opposed. With the weights as they are, Richfolk and Candlewick can block a coalition formed by the other four. If you take one vote away from either of them, that wouldn't be true any more."

"True. But what about the —" There was a knock on the door. It opened to reveal the angry face of Charlie Hogg, the representative from the Porkney Isles.

"Mr. President, this farce cannot continue any longer!"

"Calm yourself, man. Which farce?"

"Your so-called democratic voting system. The way the votes are weighted, the Porkney Isles have no power at all!"

President Perks looked puzzled. "But you have one vote to cast, in proportion to your population. Slurrey, with a slightly larger population, also has one vote. So you actually possess *more* power than Slurrey."

"No, Mr. President, we do not. The Porkney Isles, like Slurrey and indeed Fiddlesex, possess *no power at all*. The outcome of any vote is decided entirely by how the three largest districts vote. That's because any two of them together amount to a majority."

"Say that again."

"In any vote, at least two of the three largest districts will vote the same way. Their combined votes will be at least as great as those possessed by Richfolk and Candlewick, namely 16. That's a majority. You would get exactly the same result, on *any* vote — no matter how the votes split — if the Porkney Isles, Slurrey, and Fiddlesex had *no votes whatsoever!* As I said, we are powerless!"

The President thought about this. "I see your point. What do you expect me to do about it?"

"Give us another vote! Then at least the three of us could join with Sheepshire to produce a split vote. Or, if you gave Slurrey a second vote too, we could form a winning coalition!" See **TABLE 2**.

TABLE 2 Amended voting weights

district	number of votes
Sheepshire	10
Richfolk	9
Candlewick	7
Fiddlesex	3
Slurrey	2
Porkney Isles	2

"The total number of votes would then be 33," said Penny. "So 17 or more would win. I see. A coalition of Fiddlesex, Slurrey, the Porkneys, and Sheepshire could force a win."

"Yes! Any one of the smallest three districts could swing the vote — they'd hold the balance of power!"

The President pushed his chair away from his desk. "Gerry: can the Boundary Commission redraw the districts so that Slurrey and the Porkneys get an extra vote?"

The Boundary Commission Liaison Officer, Gerry Mander, shook his head. "Might swing it for Slurrey," he said. But the Porkneys are an island in Boondoggle Bay, separated from the mainland by 120 kilometres. It'd be hard to redraw *that* boundary without the populace noticing."

"Sorry," said the President.

"My constituents won't be pleased," said the representative of the Porkneys in a menacing tone.

"No. But, as you yourself have said, that will have no effect on any vote, because your district is powerless," the President pointed out. "Don't make threats you can't keep, Charlie" he said mildly.

"You can't be happy when three of the districts of Blockvotia can between them put you out of office."

"An interesting point, Charlie."

"You're the President, Freebie. There must be something you can do."

Perks studied the totals again. "I could give two extra votes to Sheepshire," he said. "The Boundary Commission could swing that, couldn't it, Gerry?"

"Certainly, sir. The district line is very wiggly along the course of the River Wastedump. We could 'rationalise' it easily."

"But extra votes for the biggest district can't possibly help the smallest one gain a share of power!" wailed the representative from the Porkneys.

"On the contrary," said the President. "If Sheepshire has two more votes (TABLE 3), you again get a share of the power."

TABLE 3 Third choice of voting weights

district	number of votes
Sheepshire	12
Richfolk	9
Candlewick	7
Fiddlesex	3
Slurrey	1
Porkney Isles	1

"Yes, look: the same coalition musters 17 votes out of 33; and again each of the three smallest districts can claim to hold the balance of power," said Penny.

Charlie looked puzzled. "That's weird," he said. "You give more power to Sheepshire, and some of it miraculously rubs off on the Porkneys."

Penny gave a humourless laugh. "No, Charlie, we don't give Sheepshire more power — we give it more *votes*. As you yourself have already argued, those aren't the same at all."

The President suddenly looked very unhappy. "I wish you hadn't pointed that out, Penny," he said. "Now you've got me very worried. If power isn't votes, what is it? I need to know," he continued, "because it's power that wins elections. Or loses them. I want to know where the power resides, what the pressure-points are."

"Who to bribe or threaten," said Charlie.

"I didn't hear that," said the President. "But if you say it outside this room, I will, even if you're in a soundproofed bunker a hundred metres underground."

"I think, sir," said Penny quickly, "that what you need is the Banzhaf power index."

"Who's he? What's that?"

"John F. Banzhaf III was a lawyer at Georgetown University in the US. He proposed a way to measure the power of representatives in a weighted voting system such as ours. The idea is that a representative can exercise power in only two ways: by

joining a losing coalition and turning it into a winning one, or by leaving a winning coalition and turning it into a losing one."

"Aren't those the same thing?" asked the President.

"That's correct, sir. When you join one coalition you leave the coalition formed by everybody else. So we need only think of one case — say creating a winning coalition. Let's say that a representative plays a *pivotal* role in a coalition if it will win with her, but lose without her. The Banzhaf power index for any particular representative is the number of different coalitions in which she plays a pivotal role.

"With the original weights, our voting system was what's called a [16;10,9,7,3,1,1] system. The vote needed for a majority — strictly speaking, a plurality — is 16; the individual representatives have weights 10, 9, 7, 3, 1, and 1. The Porkneys would be able to play a pivotal role only in coalitions totalling exactly 16. If the total was more, then the Porkneys would make no difference if they left; and if it was less, it wouldn't be a winning coalition. But there are no coalitions containing the Porkneys whose votes total 16, so the Porkneys' power index is 0."

"I *said* we were powerless."

"And with the President's new proposal? I guess that's a [17;12,9,7,3,1,1] system."

"The Porkneys are pivotal in any coalition that contains them and totals precisely 17 votes. There's exactly one, 12+3+1+1: Sheepshire, Fiddlesex, Slurrey, and the Porkneys. So the Porkneys' power index is 1."

"What about Sheepshire?"

"Sheepshire has 12 votes; so it plays a pivotal role in any coalition that contains it and has between 17 votes and 28: that is, 17-1+12. You can list those coalitions by trial and error (**BOX 1**). There are 18 of them, so Sheepshire's power index is 18."

"Holy tax-deductible!" said Charlie. "Their population would only be twelve times ours, but their power would be eighteen times!"

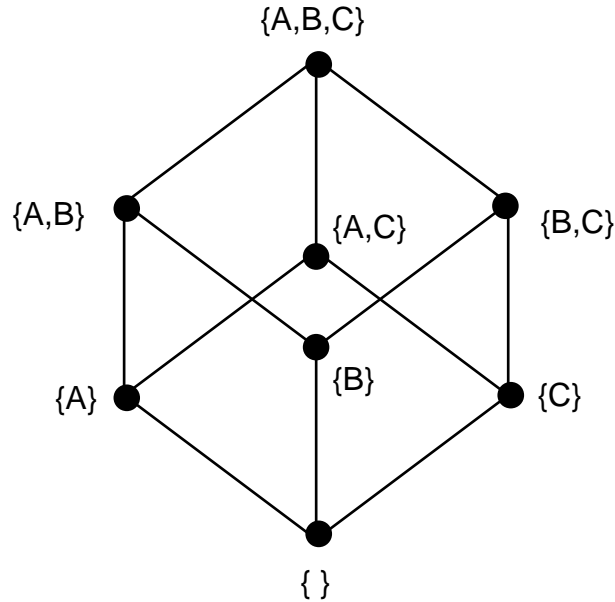
"Better than them having a population ten times as big, and infinitely more power than you," said the President.

"Is there a better way to calculate the power index than trial and error?" asked Gerry.

"Well, for a lot of representatives it's best done by computer, and even then you have to be careful," said Penny. "But there's a nice graphical method for small systems.

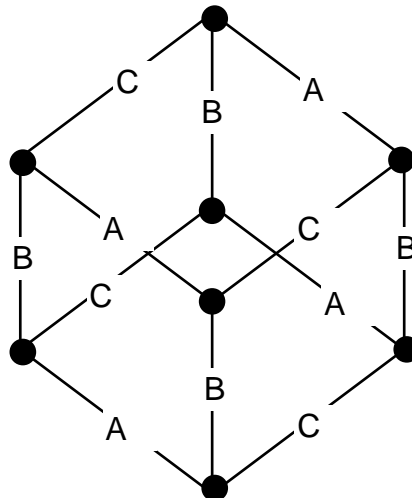
Suppose the system is $[3;2,1,1]$. That is, there are three voters A, B, and C; A has 2 votes, B and C have 1; and you need 3 for a majority.

"First, you draw a *lattice diagram* that shows all possible coalitions — all subsets of $\{A,B,C\}$ — and joins them by an edge if they differ by just one member. For three members, the lattice diagram is a cube (**Fig.1a**).



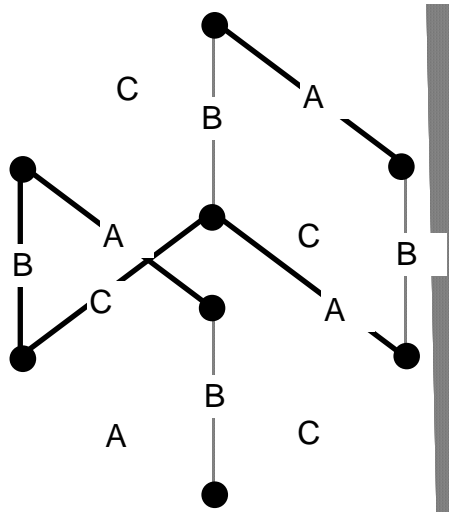
(a) Lattice diagram for three districts.

Label each such edge with the member that the sets differ by (**Fig.1b**).



(b) Labelling connecting edges.

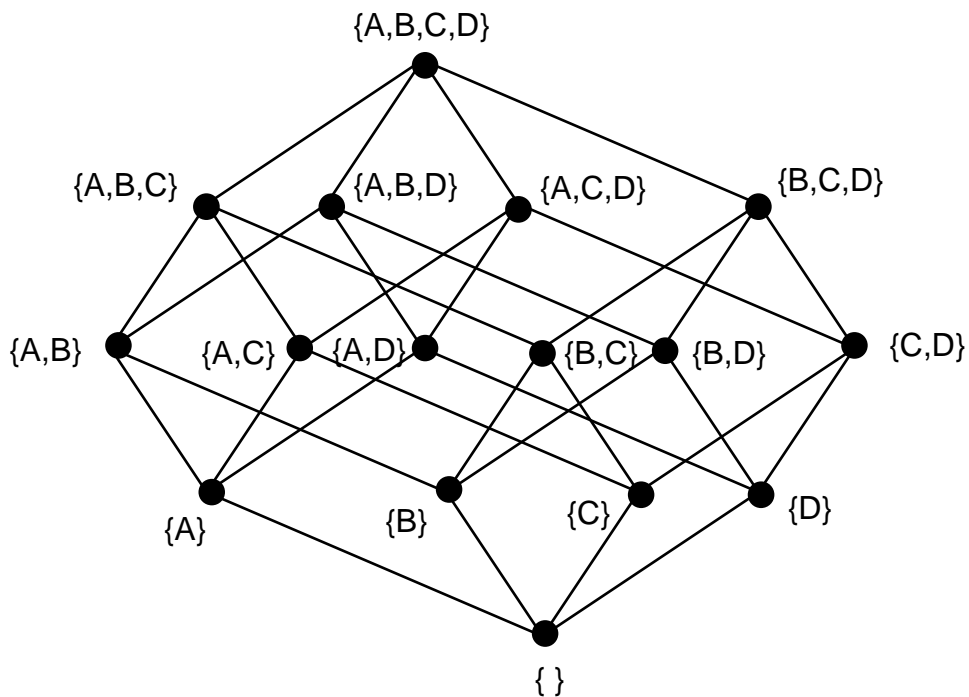
Then you mark every pivotal edge — where the total vote changes from being below the majority to being equal or above (**Fig.1c**).



(c) pivotal edges.

The power index of any given member is the number of pivotal edges with his label. Here member A appears on 3 pivotal edges, so has power index 3; whereas B and C appear on one pivotal edge each, and both have power index 1.

"You can draw lattice diagrams for more members, but they get messy. The one for four members is kind of nice, though (Fig.2)."



Lattice diagram for four districts.

The President leaned back in his chair. "Well, Charlie? What do you think?"

"I'd be happier if everybody's power index was roughly in line with their district's population," said Charlie.

"Mmmm, interesting," said Penny. "They try to do that in the States quite a lot. But it's not so easy to compute the best weights. The numbers get quite big and the computer has to try an awful lot of coalitions... Let me show you how it worked out for the Board of Representatives of Tompkins County, New York, in 1982. You can see that the power index was almost exactly proportional to the population." **(BOX 2)**.

"We could try to do something similar here," suggested Charlie.

"Maybe," said the President. His wife was a major shareholder in Blockvotia Computer Rental Inc., it could be a profitable step... A thought struck him. "Do you have any studies of the power index of the US President, Penny?"

"Yes, sir. The President of the United States has a power index 40 times as great as that of a senator, and 175 times as great as that of a member of the House of Representatives."

"That sounds wonderful."

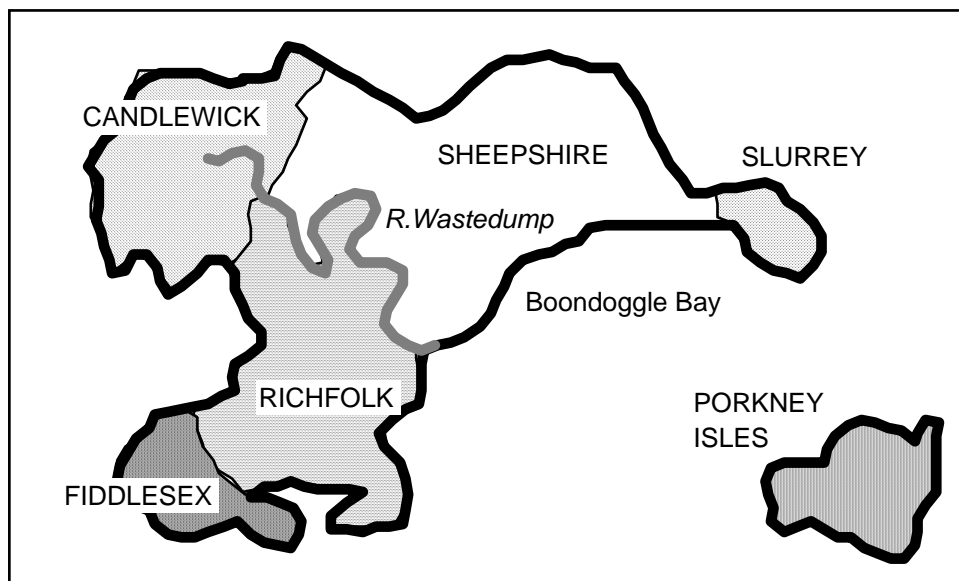
"On the other hand, the US legislative body as a whole has roughly two and a half times the power of the President."

Freebie Perks, President of Blockvotia, stared at her for a moment; then he looked Charlie firmly in the eye. "I think we'll stick to the present system," he said.

**BOX 1 Coalitions in which Sheepshire is pivotal
for a [17;12,9,7,3,1,1] voting system**

- Sheepshire+ Candlewick = 12+7 = 19
- Sheepshire+ Richfolk = 12+9 = 21
- Sheepshire+ Candlewick + Porkney = 12+7+1 = 20
- Sheepshire+ Candlewick + Slurrey = 12+7+1 = 20
- Sheepshire+ Richfolk + Slurrey = 12+9+1 = 22
- Sheepshire+ Richfolk + Porkney = 12+9+1 = 22
- Sheepshire+ Candlewick + Fiddlesex = 12+7+3 = 22
- Sheepshire+ Richfolk + Fiddlesex = 12+9+3 = 24
- Sheepshire+ Richfolk + Candlewick = 12+9+7 = 28
- Sheepshire+ Fiddlesex + Slurrey + Porkney = 12+3+1+1 = 17
- Sheepshire+ Candlewick + Slurrey + Porkney = 12+7+1+1 = 21
- Sheepshire+ Richfolk + Slurrey + Porkney = 12+9+1+1 = 23
- Sheepshire+ Candlewick + Fiddlesex + Slurrey = 12+7+3+1 = 23
- Sheepshire+ Candlewick + Fiddlesex + Porkney = 12+7+3+1 = 23
- Sheepshire+ Richfolk + Fiddlesex + Slurrey = 12+9+3+1 = 25
- Sheepshire+ Richfolk + Fiddlesex + Porkney = 12+9+3+1 = 25
- Sheepshire+ Candlewick + Fiddlesex + Slurrey + Porkney = 12+7+3+1+1 = 24
- Sheepshire+ Richfolk + Fiddlesex + Slurrey + Porkney = 12+9+3+1+1 = 26

[NOTE: This could be drawn as a set of 18 maps of the country of Blockvotia, with winning coalitions shaded. Note that the Porkneys are an island, and that the River Wastedump runs along the Sheepshire boundary and is very wiggly. Here's a suggested map (Fig 3).]



Map of Blockvotia.

BOX 2 Board of Representatives, Tompkins County 1982

Municipality	Population	Weight	Power-Index	Power/Population
Lansing	8317	404	4747	0.571
Dryden East	7604	333	4402	0.579
Enfield & Newfield	6776	306	3934	0.581
Ithaca Ward 3	6550	298	3806	0.581
Ithaca Ward 4	6002	274	3474	0.579
Ithaca South-east	5932	270	3418	0.576
Ithaca Ward 1	5630	261	3218	0.572
Ithaca Ward 2	5378	246	3094	0.575
Ithaca North-east	5235	241	3022	0.577
Groton	5213	240	3006	0.577
Caroline & Danby	5203	240	3006	0.578
Ithaca Ward 5	5172	238	2978	0.576
Ithaca West	4855	224	2798	0.576
Ulysses	4666	214	2666	0.571
Dryden West	4552	210	2622	0.576

Majority 2000 required to win.

FURTHER READING

John F. Banzhaf III, One man, 3.312... votes: a mathematical analysis of the Electoral College, *Villanova Law Review* **13** (1968) 304-332, 333-346; **14** (1968) 86-96.

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