Empires on the Moon

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Mathematics intrigues people for at least three different reasons. Because it is fun (the most important reason for inclusion in this column); because it is beautiful; or because it is useful. There are degrees of utility: a mathematical idea or method may be useful elsewhere in mathematics, it may be useful to theoretical scientists, it may be useful on the laboratory bench, it may be useful out in the world of industry and commerce, or it may be useful to ordinary citizens in their everyday lives.

I don't think that a mathematical concept has to be *directly* useful to justify its existence, or even the expenditure of taxpayers' money: mathematics is a coherent, interlocking whole and advances in one area often lead to advances elsewhere — and *those* may be useful even if the original advance wasn't. But I always take especial pleasure when a mathematical idea that at first sight seems totally useless turns out to have direct practical utility. Such examples are the best arguments against trying to judge mathematics in terms of superficial appearances. They are among the reasons why such exercises as the 'golden fleece awards' for useless science are often superficial, foolish, and misguided.

The April 1993 column 'The rise and fall of the Lunar m-pire' is a case in point. On the face of it, it was just a bit of harmless fun. Let me remind you of the problem. Earth has been carved up into separate nations, each owning one connected region of territory — land and sea. Moreover, each Earthly nation has annexed a connected region of the Moon, to create an empire that consists of two connected regions: one on Earth, the other on its satellite. Between them, these regions cover both worlds completely. What is the smallest number of colors that will color a map of *any* such disposition of territory, in such a manner that both countries in any particular empire receive the same color, but no two adjacent regions receive the same color — either on the Moon or the Earth?

The answer is unknown: it is either 9, 10, 11, or 12. It's a fun problem, but a highly artificial one.

A typically useless product of ivory tower intellectuals?

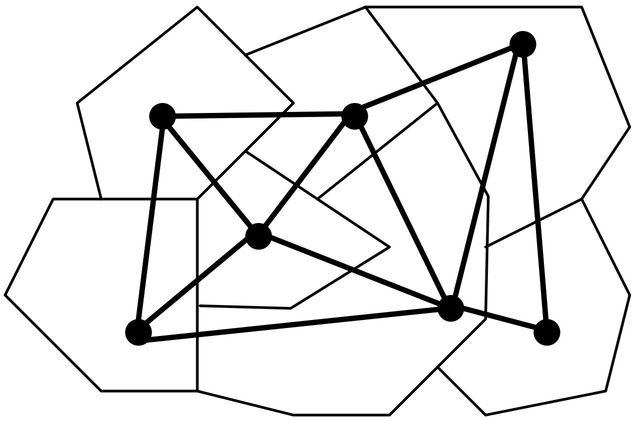
Not at all.

In October 1993 Joan P. Hutchinson of Macalester College, St. Paul, Minnesota published a thorough survey of such questions: 'Coloring ordinary maps, maps of empires, and maps of the Moon', *Mathematics Magazine* vol. 66 No.4 pp.211-226. In one section of the article she described an application of Earth/Moon coloring to the testing of printed circuit boards, discovered by researchers at AT&T Bell Laboratories, Murray Hill. The connection is not at all obvious, but it is easy to understand, and it involves some concepts that will interest recreational mathematicians and which, in any case, deserve to be more widely known. The main one is the so-called 'thickness' of a graph. This month, I'll remind you about maps, empires, and graphs, and explain what 'thickness' is. Next month, we'll take a look at the application to electronic circuit boards.

A map is an arrangement of regions, either in the plane or on a surface such as a sphere. Each region is a single connected portion of the plane or surface, and the regions make contact along common boundaries, which are curves. Often we make additional assumptions — for example that no region completely contains another region.

A graph is a diagram formed from a number of blobs, called *nodes* or *vertices*, which are joined together by a number of lines, known as *edges*. Graphs are simpler and more abstract than maps.

However, any map can be represented by assigning a node to each region and joining two such nodes by an edge if and only if the corresponding regions share a common stretch of border (**Fig.1**).



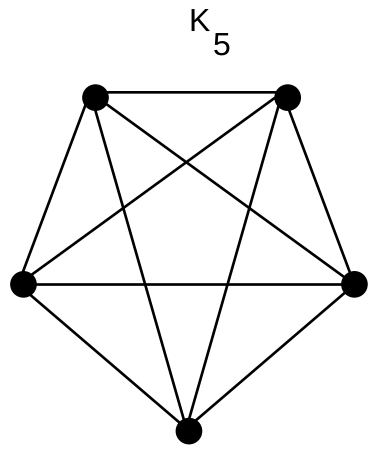
A map and the corresponding graph.

Imagine the nodes as capital cities, and the edges as highways that join cities in adjacent countries, crossing at their common border. This is the *map graph*. It represents which regions share a common boundary with others, but removes from consideration various distracting complications, such that the shapes of the regions. For many questions, the shapes don't matter, and it's often easier to get rid of them altogether — hence the map graph.

A grap is said to be *planar* if it can be drawn in the plane without any edges crossing. If we start with a map in the plane, then its map graph is obviously planar. More surprisingly, if map is drawn on the the sphere, or on several disconnected planes and spheres — as is the case for Earth/Moon maps — then the resulting graph is *still* always planar. To see why, imagine a map drawn on a sphere. Put a node in each region and whenever two regions have a common boundary, connect the corresponding nodes with edges. The result is a graph that can be drawn on a *sphere* without any edges crossing. However, any such graph can be opened up and spread out on a plane. To do this, imagine cutting a small hole in the sphere, which does not meet any of the nodes or edges of the graph. Now imagine that the sphere is made from elastic sheeting. You can pull that tiny hole, making it bigger and bigger. The rest of the sphere stretches and deforms, carrying the graph with it. By pulling it far enough you can flatten it out into a disk. Lay the disk on the plane, and you've now drawn the map graph on a plane without any edges crossing.

If the map is drawn on several spheres, we just do the same for each of them, and lay all the resulting disks out in the same plane without overlaps. The resulting graph will be disconnected — it will fall into several separate pieces, one for each sphere — but that's quite a common feature of graphs, and is allowed by their definition, so it doesn't matter.

An important graph for this column is the *complete graph* K_n, which has n nodes, an an edge joining every pair of distinct nodes. **Fig.2** shows K_5. If n is 5 or larger, then the graph K_n is *not* planar.



The complete graph K_5, which is not planar.

A map (on a plane, sphere, several spheres, whatever) is said to be k-colorable if its regions can be colored, using no more than k colors, so that regions that share a common boundary curve receive different colors. (Regions that meet only at a point, or finitely many points, can if necessary receive the same colour.) The analogous property for a graph runs along very similar lines. A graph is k-colorable if its nodes can be colored, using no more than k colors, so that nodes joined by an edge receive different colors. It is easy to see that a map is k-colorable if and only if its map graph is k-colorable. Just color each capital city, each node of the graph, with the color of the corresponding country.

The smallest such k is called the chromatic number of the graph: it tells us the minimum number of different colors needed for that graph — hence also for the corresponding map, if it is a map graph. Evidently K_n has chromatic number n, because each node is joined to every other node, so no two nodes can be colored the same.

Coloring problems have been the object of mathematical study for about a century. The best known result is the famous Four Color Theorem, which says that every map in the plane can be 4-colored. Percy Heawood proved long ago that every plane map can be 5-colored: the number was reduced to four in 1976 by Kenneth Appel of [affiliation TK] and Wolfgang Haken of [affiliation TK] in a tour de force that combined mathematical analysis with extensive computer searches and calculations. To this day, no proof that avoids heavy use of computers is known. Many genearlisations have been studied too, among them the Earth-Moon maps that I mentioned at the start of this article.

A problem closely related to Earth/Moon maps was introduced by Percy Heawood in 1890. The problem is set on the Earth only, but now each country is part of an empire containing a maximum of m countries, and the same color must be used for every country in a given empire, again with adjacent regions having different colors. (Countries in a given empire are assumed not to touch each other.) Such a map is punningly known as an m-pire. Heawood proved that an m-pire can always be colored with 6m colors, for all m _ 2.

Since an m-pire is a particular type of map, it has an associated map graph with one node per country. However, it is no longer true that every legal coloring of the map graph corresponds to a coloring of the empire. The reason is that the standard coloring rules for a graph fail to fulfil the requirement that nodes from the same *empire* receive the same color. It is difficult to handle this condition using the map graph. Instead, the construction of the graph is modified so that the coloring rules are automatically correct.

Here's how.

The *m-pire graph* associated with a given m-pire map has one node for each empire (not one for each region). If you find this confusing, think of the node as representing the emperor. Two nodes are joined by an edge if and only if the corresponding empires include at least one pair of adjacent countries. You might think of the m-pire graph as the 'invasion graph' of emperors whose empires can go to war across a common border. One node per emperor, one edge for every possible two-sided war.

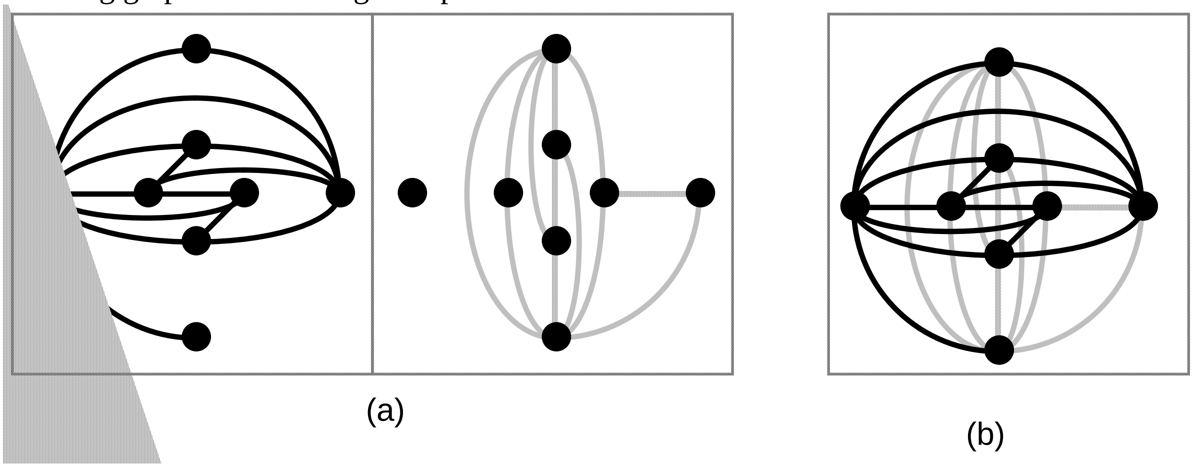
Conceptually, the m-pire graph is obtained from the ordinary graph by identifying all the nodes in a given empire — drawing them in exactly the same place. This construction often leads to multiple edges — two nodes joined by several edges instead of just one. Superfluous edges of this kind are removed, to leave just one edge.

Identifying all the nodes in a given empire automatically forces them to receive the same color, so the number of colors needed for an m-pire is the same as the chromatic number of its m-pire graph.

In 1983 Brad Jackson (San Jose State U) and Gerhard Ringel (U California, Santa Cruz) used this approach to prove that the number 6m in Heawood's theorem cannot be reduced. They did this by demonstrating that you can find an m-pire whose m-pire graph is the complete graph K_{6m}. Since K_{6m} definitely needs 6m colors, there is a m-pire that cannot be colored with less than 6m colors.

There are connections between Earth/Moon maps and m-pire maps. In fact, an Earth/Moon map can be viewed as a particular kind of 2-pire, with a slightly curious underlying geometry (two spheres) which splits all the 2-pires into two pieces. Its graph consists of two disjoint planar graphs — for example, one possible arrangement is shown in **Fig.3a**. (The rounded shape has nothing to do with the Earth or Moon: recall that any graph on a sphere, or several spheres, can be deformed so that it lies in a plane. It's just easier to show the shape of the graph here using curved edges.)

Suppose that we now think of this Earth/Moon graph as a 2-pire graph, so that nodes belonging to the same empire are identified to create **Fig.3b**. We see that the resulting graph need no longer be planar. Indeed this one isn't.



(**a**) Graphs for the terrestrial and lunar territories of a set of eight empires.

(**b**) Identifying corresponding nodes to create the corresponding 2-pire graph.

However, the graph is 'almost planar'. The way it is constructed shows that its edges can be separated into two subsets, each of which forms a planar graph on the original set of nodes. Here the two subsets are the edges in Fig 3a and those in Fig.3b.

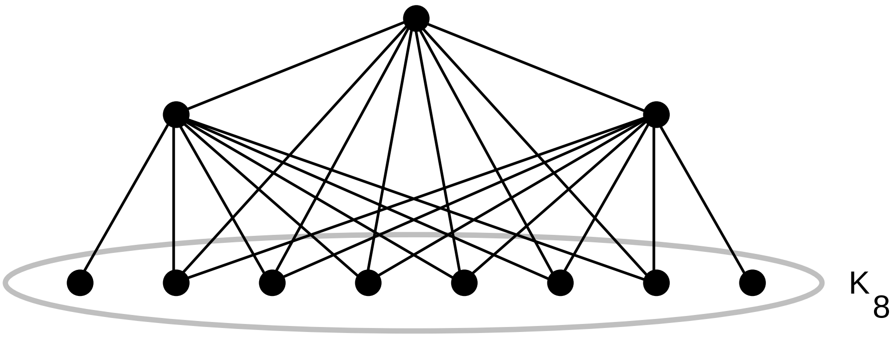
Such a graph is said to have thickness two. In general, a graph has thickness t if its edges can be separated into t subsets, and no fewer, in such a manner that each subset forms a planar graph. Now, every map graph is planar, even when the map lives on a sphere. An Earth/Moon map is made up from two separate planar maps: one on the Moon, the other on the Earth. Each empire is represented exactly once in either of these maps. So every Earth/Moon graph has thickness two: one planar bit for the Earth part, the other for the Moon part. The converse is also true: every graph of thickness two corresponds to an Earth/Moon map (although the territories involved may not completely cover the two worlds: there may be regions unclaimed by any of the empires).

Because an Earth/Moon graph is a special kind of 2-pire graph, Heawood's theorem implies that 12 colors are *sufficient* for any Earth/Moon graph. However, we can't conclude directly that 12 colors are also necessary. The reason is that not every 2-pire correesponds to an Earth/Moon map. In an Earth/Moon map, each empire has one region on the Moon and one on the Earth. If we think of this as a 2-pire, then the regions form two separate 'islands', and there is exactly one region from each empire on each island. In contrast, a 2-pire consists of a number of pairs of regions, which need not be arranged to form two islands — and even if they are, some empires might have both territories on the same island.

In fact, *none* of the known 2-pire graphs that actually require 12 colors can be turned into Earth/Moon maps. It therefore remains possible that *fewer* than 12 colors might always be enough for an Earth/Moon graph.

For instance, the complete graphs K_9, K_{10}, K_{11} and K_{12} are all 2-pire graphs, but they have thickness 3, and so cannot be Earth/Moon graphs (because those have thickness two). In fact, the thickness of K_n is 3 if n = 9 or 10, and is the greatest integer not exceeding (or 'floor' of) (n+7)/6 otherwise.

Fig.3b is in fact the complete graph K_8, so K_8 has thickness 2. This means that it can be represented as an Earth/Moon graph. This proves that at least 8 colors are needed in the Earth/Moon problem. Rolf Sulanke (Humboldt U Berlin) increased this lower limit to 9 by showing that the graph of **Fig.4** has thickness 2 and chromatic number 9.



Sulanke's graph of thickness two, which requires nine colors.

The concept of thickness, then, is the deep mathematical idea that underlies the recreational puzzle of Earth/Moon maps. You might like to think about Earth/Moon/Mars maps, where every emperor has *three* territories, one on each world. These maps are particular kinds of 3-pire map, and their 3-pire graph always has thickness three. In general a graph of thickness t can be thought of as the t-pire graph of a system of galactic empires on a collection of t planets.

Map-coloring problems of this kind are great fun — but they have little obvious practical significance. Even if we had planetary empires, the geographers could always color their maps by trial and error — and in any case they might not want to follow our coloring rules. However, we shall see next month that there *are* applications of the concept of thickness; however, they are not literal translations of the 'map' image. Instead, they apply to the testing of electronic circuits.

Mathematics is abstract and general: the same idea has many realisations. Some are more fun than others — and some are more practical than others.