

Gambler's Ruin at the Casino Incroyable

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The tall, silent Englishman in the impeccable white suit was suddenly bored. He scooped up his winnings from the roulette table, casually tossed a thousand-franc chip to the croupier, and made his way to the bar of the Casino Incroyable.

"Monsieur?" said the barman.

"The usual."

"Oui, monsieur."

The Englishman reached out a manicured hand and grasped the barman's arm. "Three measures of Gordon's, one of vodka, half a measure of—"

"Oui, monsieur," said the barman tiredly. "Shaken, not stirred. How could I forget?"

"With a large slice of lemon-peel, so thin as to be translucent."

"Of course, monsieur. Marie will bring it to your usual table on the balcony?"

"Uh — no, not tonight." Out of the corner of his eye the Englishman had caught a glimpse of a tall, slim brunette whom he recognised as a minor movie celebrity. And her escort — why, without the theatrical moustache and the blond wig he would be the spitting image of *Le Zéro*, the elusive international spymaster! He loved it when a mission came together. "No, tonight you will serve my drink at —" he flicked a glance over his shoulder to see where they were heading — "the craps table." He laid a crisp fifty-franc note on the bar and glided silently in pursuit of his quarry.

Little did he realise that this was to be one of the most dangerous missions in his entire career.

The barman turned to Marie, the cocktail waitress, and shrugged. "We are still out of dry martinis? Yes. Then dump half a can of 7-up in a champagne glass and stick a lemon in it, as usual."

When the Englishman arrived at the craps table, the brunette was holding the dice. There was an empty seat between her and *Le Zéro*, and he sidled expertly into it. "You look lucky tonight, my dear," he said, placing a stack of chips in the 'line' area of the table — a 'pass line' bet indicating that he expected her to win. "Please excuse my boldness, but I recognise you from the newspapers. Your name is O'Hara, and you are rapidly becoming a movie celebrity."

"That's right," she said, surprised. "Starlet O'Hara, that's me. And you?"

"My name is Blond. James Blond."

She picked up the dice, and rolled. Two ones stared up at them. "Snake eyes," she said in disgust. "Not as lucky as you thought." Blond watched as the house took his chips. "Your turn to be shooter," said O'Hara.

"No, I left my Beretta in the hotel room."

"I mean, your turn to throw the dice."

"Very well." Casually, Blond shook the dice — being careful not to stir them — and tossed them along the table so that they hit the far end and bounced. "Natural." Blond had thrown a seven — an immediate win. He let the stake and winnings stand, and threw again. Two sixes. "Crapped out. Easy come, easy go." He passed the dice to the next shooter, the man he had recognised as *Le Zéro*.

His mind went back to the previous day's briefing. According to M's best information, *Le Zéro* ran more than a hundred agents for SMASH, a murky underground organization specializing in political assassination, nuclear blackmail, the international narcotics trade, and counterfeit mashed potato. Starlet O'Hara obviously didn't realise that she was in mortal danger.

Fortunately *Le Zéro* would never penetrate Blond's playboy disguise.

"Thank you, Mr. Blond," said *Le Zéro*. "It is not often that one has the pleasure to play craps with a member of the British Secret Service."

Blond decided upon candour. "How did you recognise me, *Le Zéro*?"

"It was very unsubtle of you to score 007 on your first throw of the dice."

"It was also very unsubtle of you to wear a moustache that is immediately recognisable as one of those sold by Hutchinson's Theatrical Costumiers of Worbold Street," riposted Blond.

Le Zéro sneered at him. His left hand strayed to his jacket pocket. "Unlike you, Mr. Blond, I have not left my gun in the hotel room. But I will spare your life, and permit Mlle. O'Hara to retain her honour and her freedom, if you can demonstrate your skill at craps."

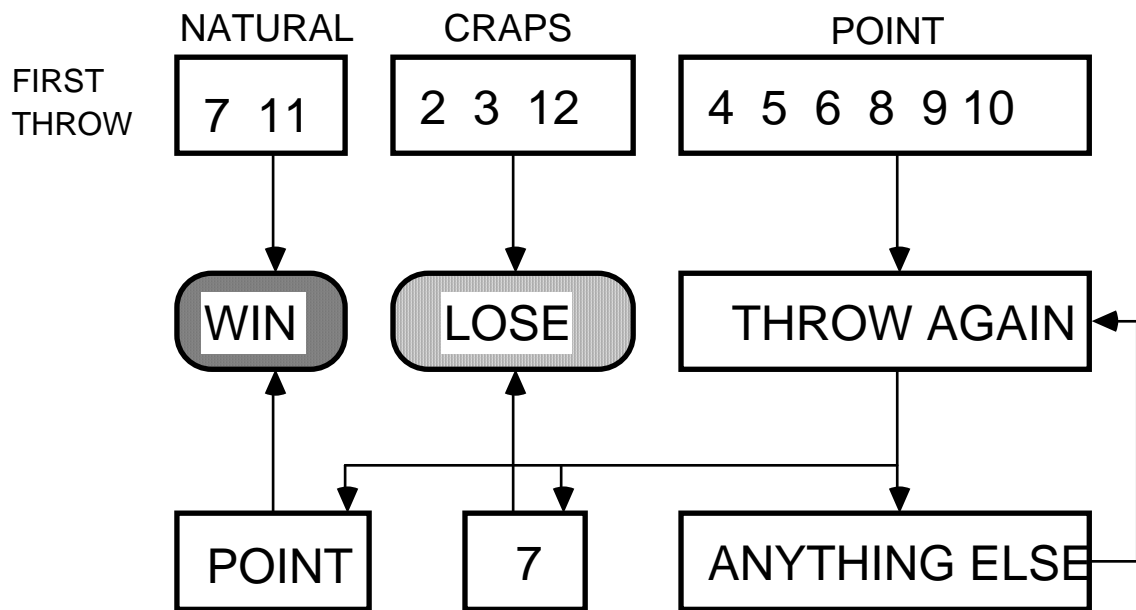
The starlet gasped. "What do you mean? Retain my freedom? You told me you could get me into something really big."

Blond laughed humorlessly. "Unfortunately, O'Hara, you've fallen in with *Le Zéro*, one of the most dangerous men that walks the face of the earth. However, he told you the truth: the harem of the Potentate of Jumbostan *is* unusually large."

O'Hara turned white as a sheet. *Le Zéro* smiled. "How perceptive of you, Mr. Blond. Now, I have arranged with the Casino Incroyable for you to have free credit of 2

million francs. You may make only pass line bets, gambling upon my skill as shooter. I will retain the dice throughout. If you can win enough to increase that to 5 million francs, then you and the girl may go free. If you lose it all, then you will owe the casino 2 million francs. Since you cannot pay such a sum, I assure you that you will not survive the night — and Mlle. O'Hara will be inside a crate on the next flight to Jumbostan."

Blond mentally reviewed the rules of Bank Craps, otherwise known as Las Vegas Craps, the casino version of the American dice game in which all players' bets are made with the bank, not against each other (see Fig.1).



The rules of craps.

The shooter — the player who was currently rolling the two dice — could win by throwing a total of 7 or 11 on the first roll, or lose by throwing 2, 3, or 12. If any of the remaining numbers, 4, 5, 6, 8, 9, or 10 came up, then that number became the shooter's 'point'; and in order to win, the shooter must continue to throw until either the point is thrown again (a win) or the shooter threw a 7 (a loss). All players placed bets against the bank, the commonest being a 'pass line' bet (that the shooter will win). The payoff odds for such a bet are one to one: every franc bet gives one franc in winnings plus the return of the stake.

Blond made some rapid calculations. He knew that in craps the probability that the shooter will win is exactly 244/495, or approximately 0.493. Therefore the probability of winning a pass line bet is 0.493 — very close to evens, but a slight disadvantage.

Poker-faced, Blond placed a million-franc chip in the pass line area. O'Hara drew her breath in sharply. "But that is half your stake!" She swallowed nervously. "You are a very bold gambler."

Blond leaned close. "Not really," he said. "A bold gambler is one who bets his entire stake — or as much of it as is necessary to reach his target. I should have bet 2 million francs."

"Why?"

"Because in 1965 L.Dubins and L.J.Savage proved that when the odds are against you, as they are here, you maximize your chance of winning by playing boldly. The basic idea is that the longer the game is drawn out, the less chance you have of winning. Timid play — placing small bets — prolongs the time taken to lose, but makes losing more likely."

"Oh. But then, why did you bet only 1 million francs?"

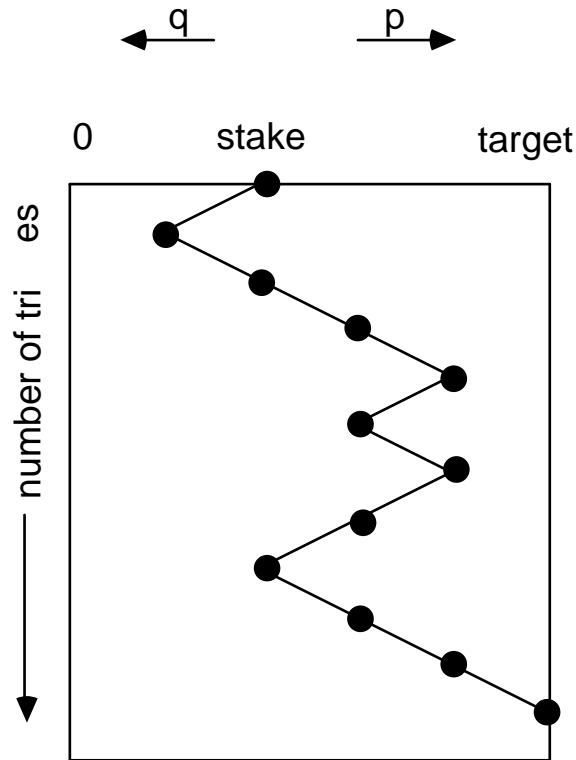
"Because that is the house limit. They won't accept a larger bet."

"I see."

"So instead of bold play, I am going to bet the house limit every time. Which means that I can calculate my chances of winning using the theory of the gambler's ruin."

"I don't like the sound of that," said O'Hara.

"Suppose that you always bet the same amount, which we will call one unit. Assume payoff odds of 1-1. Suppose that p is your chance of winning one bet, and let $q = 1-p$ be the chance of losing one bet. Your progress forms a *random walk* in which you have a probability p of moving one unit in the positive direction, and q of moving one unit the other way (**Fig.2**).



Gambler's ruin seen as a random walk. At each play, the gambler's current stake either increases by 1 unit with probability p , or decreases by 1 unit with probability $q = 1-p$. In the long run either the gambler reaches his target, or loses his entire stake.

So you follow a kind of zig-zag path. Eventually this either hits the barrier at 0, when you are ruined; or you attain your target — the amount you want to accumulate — and stop.

"No, suppose that your initial stake is s , and let your target be t . Then according to the probabilist William Feller, the probability that you will reach your target *before* going bust is

$$P(s,t) = (1-(q/p)^s)/(1-(q/p)^t)$$

if $p \neq \frac{1}{2}$; and

$$P(s,t) = s/t$$

if $p = \frac{1}{2}$.

"Here my unit is a million francs, my initial stake is 2 units, and my target is 5 units. So $p = 0.493$, $q = 0.507$, $s = 2$, and $t = 5$. Therefore

$$P(s,t) = (1-(.507/.493)^2)/(1-(.507/.493)^5) = .383.$$

I have a 38% chance of winning.

"Suppose instead I were to play more timidly, say by using a unit of 100,000 francs. Then $p = 0.493$, $q = 0.507$, $s = 20$, and $t = 50$. Therefore

$$P(s,t) = (1-(.507/.493)^{20})/(1-(.507/.493)^{50}) = 0.245,$$

which is worse. And if my unit was 10,000 francs then $p = 0.493$, $q = 0.507$, $s = 200$, and $t = 500$. Therefore

$$P(s,t) = (1-(.507/.493)^{200})/(1-(.507/.493)^{500}) = 0.000224,$$

which is *much* smaller. So it really does pay to be bold, you see."

Starlet O'Hara stared at Blond in wide-eyed adoration. "You're so clever."

Blond puffed out his manly chest. "Yes, but a 38% probability isn't too good. I need to be lucky, as well." He relaxed. "Fortunately, I always am." He toyed idly with a button on his shirt, which a skilled eye would have noticed was slightly larger than the others.

Le Zéro threw a 4. Blond and O'Hara watched as he tried to make his point. But he threw a 7 first. Blond shrugged, and placed his remaining million francs on the pass line. O'Hara gripped his arm in terror. Bond continued to fiddle with his shirt button. *Le Zéro* threw a 5. It took him eight more attempts to throw it again and make his point: fortunately he did not throw a 7 first. Blond, grinned: his luck had started to change. He picked up one million francs of his winnings and let the other million ride...

An hour later, Blond and O'Hara walked arm in arm from the casino.

"*Le Zéro* was very annoyed," said O'Hara. "We are still not safe, I think."

"Oh, I don't know about that," said Blond. Little did the international spymaster realise that when Blond had reached his target 5 million, and had contemptuously tucked a million-franc chip into *Le Zéro's* suit pocket, he had also affixed a small remote-controlled incendiary bomb — which Q had cleverly disguised as a paperclip — to *Le Zéro's* lapel.

Blond looked wryly at his watch, into which Q had built the detonator. "Damn, the date's not right." He gripped a tiny knob at one edge and twisted.

"*Le Zéro* will be livid," said O'Hara.

From behind them came a loud bang and screams. "I imagine he's *fuming*," said Blond. "Fancy a bottle of Bollinger '57 up in my room?" O'Hara leaned over and whispered acceptance in his ear.

Now, as I said at the beginning, little did Blond realise that this mission was to be the most dangerous of his entire career. Up till that point it had been pretty much routine. Q's patent remote telekinetic unit for dice, cunningly disguised as a shirt-button,

had performed brilliantly; there had never been any danger except when O'Hara had exposed a little too much cleavage and Blond's suddenly sweaty fingers had nearly caused a shirt-circuit. But now, with his guard relaxed, Blond was about to discover that the evening still had some surprises to offer...

Starlet O'Hara lay back on the king-sized bed with a glass of champagne in her hand, giggling uncontrollably. Blond sat down beside her, sipping at his Bollinger '57. Slowly O'Hara unzipped the side of her dress, while Blond's eyes popped out. Then, with the speed of a rattlesnake, she pulled out something small and deadly and pointed it at him.

Blond froze. He recognised the weapon immediately.

It was an Income Tax Demand Form.

He struck his forehead with one hand. "I should have realised when you agreed to come up to my room. You're a government agent."

"That's right, Blond. By day, I am a Trainee Inspector, Third Grade, at the Inland Revenue."

"But by night you become *Taxwoman!* Feared throughout the civilized world. This was all a set-up."

"No, I was originally after *Le Zéro*. He claimed ten per cent too much on a taxi fare in 1983. But when he arranged your line of credit it occurred to me that I could shake two birds with one stir. Wooops, this champagne is going to my head."

Blond began to protest. "But gambling winnings aren't taxable!"

"Correct. But you were given free credit. That has to be worth *something*. And whatever it's worth, it's taxable."

"*What?*" cried Blond in astonishment.

"The Zarin case. In 1980 David Zarin, a compulsive gambler, was given virtually unlimited credit at a casino in Atlantic City, and ran up a debt of \$3 million. According to New Jersey law, his debt to the casino was not legally enforceable, and the casino settled out of court for a much smaller amount. But then the Internal Revenue Service — my American counterpart — claimed taxes on the remainder of the debt, on the grounds that a negative debt is equivalent to income. What's sauce for the US is sauce for the Englander, Blond."

Blond shook his head, unable to believe his ears. Even Miss Manypunny had never made a pun as bad as that. "But I saved you from a fate worse than death in the harem of the Potentate of Jumbostan!"

O'Hara indicated that she understood this point. "However, that doesn't affect your tax liability." Then her expression softened. "Unless you want to gamble again. When the Zarin case came to court, there were a lot of legal arguments about what \$3 million gambling credit was really worth. The Internal Revenue Service thought it was \$3 million. Zarin's lawyer claimed it was worth nothing, because his client gambled it all away and never received a single penny in income. The funny thing was, the court never did take the trouble to work out the actual worth of Zarin's free credit."

"Of course not. They're lawyers, not probabilists."

"Indeed. So here's the deal. I bet you can't tell me what a *large* amount of free credit is worth. Not just 2 million francs — a billion, or a trillion. Assuming you keep making million-franc pass line bets. Get that right, and I'll forget I ever heard *Le Zéro* offer you any credit, and thank you for saving me from the clutches of the Potentate of Jumbostan."

Blond's head started to ache, but he ignored it. This was the worst fix he had ever been in in his life. Fortunately he was an expert probabilist, fully trained in the exotic arts of *chi-square* and *histo-gram* by the famous Probable Arts *sensei* Martin Gale. Blond had finished in the top percentile at the 1993 Bernoulli Trials, and would have won except for a stupid oversight in the Student's t-test.

"I'll give you a hint," said O'Hara unwisely. "The value of a credit line is the maximum expected profit you can make."

"Ah," said Blond. She had given him just enough advantage to seek out his opponent's weak points. *I'm a lean mean computing machine*, he told himself, trying to get properly psyched up for the battle. He began to circle slowly round the problem, alert for its every move. "Let me see. I start with s units and keep making one-unit bets until either I reach t units or I go bust. In the former case, my profit is $t-s$ units; in the latter it is 0. So —" he grasped the problem by its corollaries and attacked with a breathtakingly rapid Feller Formula — "the expected profit is given by

$$E(s,t) = (t-s)P(s,t) = (t-s)[(1-(q/p)^s)/(1-(q/p)^t)].$$

I believe I have drawn first blood."

O'Hara watched open-mouthed as James Blond, secret agent 007, grappled with the most inscrutable opponent of his entire career.

"Ah!" gasped Blond, instinctively sensing a line of attack despite the confusing flurry of symbols being rained upon him. "I have to choose t so as to maximize $E(s,t)$. And all I know is that I have virtually unlimited credit." Of course! He could slice

through its defences with an asymptotic approximation! "Suppose I let t become very large, and let s also become large, so that $t-s = d$ is fixed. Let $r = q/p > 1$. Then

$$P(s,t) = (1-r^s)/(1-r^t)$$

which tends to r^{-d} as t tends to infinity. Then $E(s,t)$ tends to dr^{-d} ." Now, a quick thrust of the calculus to nullify the function's derivative... "This takes its maximum value when $d = 1/\log(r)$. How am I doing?"

"Surprisingly well," said O'Hara, impressed despite herself. Blond was surviving far longer than she had anticipated.

Blond made a powerful substitution. "It follows that for large s , the target increment that — ugh! — maximizes expected payoff is approximately $t-s = d = 1/\log(r)$. Then as s tends to infinity, we have

$$P(s, s + 1/\log(r)) \sim r^{-1/\log(r)} = e^{-1}$$

and

$$E(s, s + 1/\log(r)) \sim (e \log(r))^{-1}$$

where $e = 2.71828\dots$ is the base of natural logarithms."

Now he had the problem by the throat, it would not escape him now. "For pass line bets at craps," he gasped, twisting its hypotheses cruelly behind its back, " $p = .493$, and $r = .507/.493 = 1.028$. Therefore $\log(r) = .028$, and $d = 1/\log(r) = 1/.028 = 36$. So my optimum choice is to set myself a target that is —" one final thrust would finish the battle — "36 units more than my stake!" He collapsed, covered in sweat. The probabilistic calculation lay inert on the floor.

"Excellent," said O'Hara. "You surprise me, Mr. Blond. Though you did leave at least one lemma unproved."

"Ah, but that one was trivial. All you have to do is—" But she was trying to distract him. Why? Of course, the problem was only *pretending* to be dead! What had he forgotten? What had his *sensei* told him? Ah, yes: "Always read the question." There was a final step still unansw— without warning the problem reared up on its hind legs and leaped at him, spitting streams of obscure mathematical symbols, but Blond was ready for it. Ducking the poisoned edges of a violently hurled ninja asterisk he reduced the problem to its lowest common denominator, and squeezed...

"Come on, you wretch! What's my expected payoff?"

They wrestled around the floor, smashing a large number of valuable antiques and tearing to shreds a priceless first edition of Laplace's *Theorie Analytique Des Probabilités*. Blond used knees, elbows, and as a last resort did something extremely

painful with a table of logarithms. The problem flopped, gurgled, and expired. "QED," panted Blond. "My expected payoff is —" he took a deep breath — " $(0.028e)^{-1} = 13$ units, or 13,000,000 francs in this case."

"13,006,305 francs, to be precise," said O'Hara. "But I won't argue over a few insignificant figures. In Zarin's case the house limit for a pass line bet at craps was \$15,000, so if he had followed your strategy he would have won, on average, about \$195,000. That's what his free credit was really worth, unless there's an even better strategy, of course. Maybe he should have tried blackjack, where a skilled player actually has an advantage. But since he lost \$3 million, I don't think 'skilled player' is a plausible assumption."

"What was the court's decision in the Zarin case?"

"It went to appeal, and an appellate court finally ruled that he owed no tax."

"You mean that struggle was all for nothing?" cried Blond, outraged at how he had been tricked.

"Not necessarily. The question has never been tested in British law, Mr. Blond."

Blond looked hungrily at her. "You said you would thank me for saving you from the clutches of the Potentate of Jumbostan."

"Oh, Mr. Blond." Starlet O'Hara licked her lips and smiled. "Thank you for saving me from the clutches of the Potentate of Jumbostan." She put her champagne glass down on the bedside table, tucked the tax demand back inside her dress, and closed the zip.

"Was that it?" said Blond.

"Count yourself lucky to get so much as a thankyou from *Taxwoman*," said O'Hara. "I will leave now, before we both do something I shall regret." She paused, and he saw her silhouette, slim and seductive in the doorway. "How do you feel about the way your evening turned out, Mr Blond?"

Blond sat groggily on the side of the bed, and glared at her. "Shaken — but not stirred," he said.

FURTHER READING

L.Dubins and L.J.Savage, *How to Gamble If You Must*, McGraw-Hill, New York 1965.

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Michael Orkin and Richard Kakigi, What is the worth of free casino credit?, *American Mathematical Monthly* **102** (Jan. 1995) 3-8.

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