

# How not to Lose at 9-in-a-row

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Bumps the goose-girl and Grimes the shepherd-boy were sitting together at the side of the valley in the spring sunshine. On the far slope, black sheep and snow-white geese mingled contentedly, while Fumble the sheepdog snoozed beneath a tree. The bees buzzed, the butterflies fluttered by, and the daisies... dazed. It was idyllic.

"I'm bored," said Grimes.

"Me too," said Bumps. "Total relaxation on a perfect day is fine — for a time. But your brain goes kind of fuzzy. I know, let's play a game."

Grimes shook his head. "No, I've played games against you before. You've got too logical a mind, you always win."

"True. Let me — aha. We'll play a game with no known strategy. That way I can't force a win by superior logic."

"OK," said Grimes, who had been so lulled by the spring sun that he wasn't thinking very clearly. Even without a winning strategy, Bumps could use her logical superiority to select some pretty good moves. "Such as what?"

"Five-in-a-row," said Bumps decisively. "Otherwise known as go-moku, go-bang, pegotty, or pegity (Parker Bros. USA). We each need some pieces, and a board."

"Nothing like that around here."

"Well, we'll just have to *improvise*." She stared across the valley. Farmer Quinn had been preparing the ground for new drainage pipes, and it was marked out into a grid of squares with white lines, like those used to mark sports pitches. "That field over there can be the board. You use your sheep as pieces, and I'll use my geese."

"It's a long way to go to move them. And they won't stay put."

"You can whistle up old Fumble and get him to move your sheep by remote control — *and* make sure they stay moved. I'll just call sweetly to my geese, they're *so* well-behaved."

This plan proved acceptable to all save Fumble, who felt that chasing sheep around a hillside on a perfect spring day was a pretty stupid idea. But when he tried to explain his point of view it came out as "woof!", which it always did, and nobody took any notice, which *they* always did.

It's a dog's life.

"What are the rules?" asked Grimes.

"We each play one piece in turn, putting it on to a square. At most one piece can go on any square, of course. The first to get five in a row, next to each other, wins."

















"Are diagonal rows allowed?" asked Grimes.

"Naturally, otherwise the game would always be drawn with perfect play," said Bumps.

"Huh?"

"Oh, sorry, I forgot for a moment that you're not a logician. I'll explain later. Let's just agree that diagonal lines are allowed, OK?"

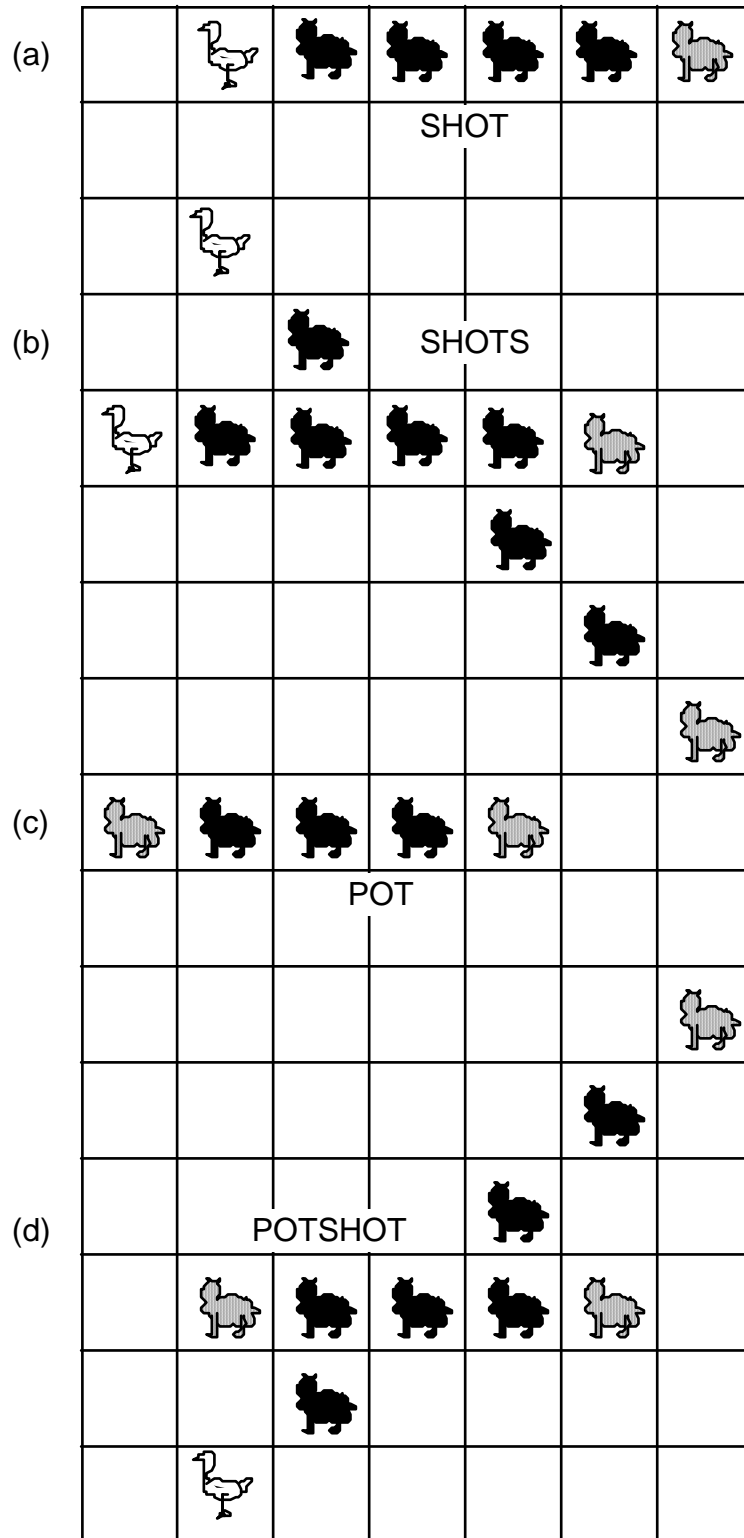
Bumps won the first twenty games. **Fig.1** shows a typical one.

					 16		
				 5	 12		
				 3	 14	 4	
	 13		 9	 1	 2		
		 11	 10	 6	 8		
			 7		 15		

A game of five-in-a-row. Bumps wins.

But Grimes was undaunted, because he was beginning to see how to improve his play. For the first few games he had ignored obvious and immediate threats to win (**Fig.2a**), until Bumps started warning him by shouting '*shot!*' every time such a position occurred, to remind him to block it. He did the same, though not as frequently. But he still ran into trouble when she created two such possibilities simultaneously (**Fig.2b**). She started warning him by yelling '*shots!*', but by the time she did, it was too late, of course. So then she had to warn him that she had reached a position that guaranteed a *shots* position on her next move (**Fig.2c**), so then she shouted '*pot!*'. And if either of them created a *pot* and a *shot* at the same time (**Fig.2d**), they shouted '*potshot!*'... and so it went.<sup>1</sup>

<sup>1</sup> This terminology is due to Elwyn Berlekamp, John Conway, and Richard Guy — see Further Reading



Winning threats. Grey sheep indicate possible winning moves. (a) Shot. (b) Shots. (c) Pot. (d) Potshot.

It made the game a lot more fun and seemed to improve their tactics. Grimes started shouting 'hot!' when he had the chance of creating a *pot* next time, and they collapsed in giggles when Bumps achieved a *hotshot-hotpot*.

Eventually the games got so long that poor old Fumble ran completely out of steam, trying to control twenty or more sheep at once, so they called a time-out while the dog collapsed under a tree.

"Bumps, you said that no complete strategy is known for five-in-a-row," said Grimes. "Why not? And what *is* known?"

"Well," said Bumps, "let's think about  $n$ -in-a-row, which has the same rules, but you win by getting  $n$  pieces in a row — horizontally, vertically, or diagonally. Roughly, if  $n$  is small then there's an easy win for the first player because the task required is not very hard anyway; and when  $n$  is large the second player can always force a draw because the first player has to give so much advance warning of his intentions that it's easy for the second player to break up any row that's forming. The difficult games are in between, and the hardest seem to be 5- and 6-in-a-row."

"Ok," said Grimes. "I can see that 1-in-a-row is a win for the first player, because he wins on the first move."

"Brilliant analysis, Grimes."

"Look, I'm just polishing off the trivial case first, OK? He also wins with 2-in-a-row, on the second move, because wherever the second player goes, there's always a place to play next to the first piece. The opening move is a *shots* position."

"Right."

"With three pieces the opening move is a *pot* position. Whatever the second player does, the first can create two in a row with both ends open. There are several different cases but it's still pretty obvious."

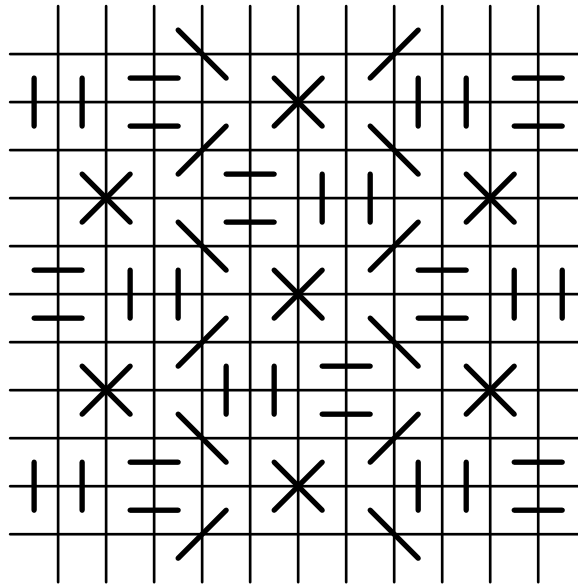
Bumps nodded, and called to one of her geese that was getting a bit too interested in a tangle of blackberry thorns. It squawked in protest, but backed away.

"I'm having trouble with four-in-a-row, though," said Grimes. "I can't decide whether the opening move is a potspot-potshot, or a whatnot."

"I'm not surprised — it's a hard problem. Carlyle Lustenberger<sup>2</sup> used a computer to prove that it's a win for the first player — provided the board is at least 4\_30. Coming from the other direction, there's a very beautiful proof that the second player can force a draw in nine-in-a-row. The idea is to pair off squares in a clever way (**Fig.3**).

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<sup>2</sup> Pennsylvania State University



Hales-Jewett pairing that proves nine-in-a-row is drawn if played properly. Adjacent squares linked by thick lines are paired. Any row of nine squares, orthogonal or diagonal, contains both members of some pair.

Whenever the first player occupies one square of such a pair, the second player immediately occupies the other. Here the pairs are arranged so that any row of nine pieces must contain *both* members of some pair. Which means that no row of nine pieces all the same colour can ever be formed, so it's a draw. This kind of strategy is called a *Hales-Jewett pairing*, after A.W.Hales and R.I.Jewett, who used the same approach to prove that five-in-a-row is a draw on a 5\_5 board. (*Readers may like to look for such a pairing. Note that the paired squares need not be neighbours.*)

"A group of Dutch combinatorialists, using the pseudonym T.G.L.Zetters, has proved that the second player can draw eight-in-a-row. Their method also suggests that seven-in-a-row should be a drawn game, but that's not completely proved. Whether or not there's a winning strategy with rows of length five and six remains open. If you're going to prove that the game is always drawn, you'll probably need a different approach from the Hales-Jewett one."

Grimes settled back against the hillside. "Yes. When we play, the biggest problems seem to be long-range effects. You put a piece a long distance out from where the action is, and twenty moves later it turns out to be crucial."

"Yes. Mind you, you could have argued like that for nine-in-a-row. Pairing strategies can deal with that kind of problem — by showing that it's irrelevant — but only if they exist."

Grimes leaned on one elbow. "Fumble looks refreshed. Let's play another game, I've just thought of a new one."

"If you want," said Bumps.

"Same kind of rules, except that to win you have to get four of your pieces in a 2\_2 square."

"Hmmm." Bumps thought about it for a moment. "I think I should warn you that —"

"No! Stop *thinking*, let's *play*! I go first!"

"You asked for it," said Bumps.

After fifty-seven consecutive draws, Fumble crawled back into the shade of the tree and Grimes flung himself to the ground in frustration. "I thought it was a good idea," he said faintly.

"It was, Grimes, but you didn't think it through properly. There's a simple Hales-Jewett pairing that proves the game is always a draw if played properly. Imagine tiling the plane with dominos, 2\_1 rectangles, arranged like bricks in a wall so that each layer is staggered relative to its neighbours (see **Fig.4**, top). Pair off squares in a domino. The second player's strategy is that whenever the first player occupies one square of a domino, she immediately occupies the other. You can check that however a 2\_2 square is placed, it must include a domino. So the strategy prevents a 2\_2 square of a single colour being formed."

"I see. What about other shapes than 2\_2 squares?"

"You've reinvented a game proposed by Frank Harary. Choose some shape formed by adjacent squares — known as a *polyomino* (a registered trademark of Solomon Golomb). Players take turns placing pieces, trying to form some copy of that polyomino in their colour. It doesn't matter if there are other pieces around, of course: just that some set of pieces of their colour is the same shape as the polyomino. Call the polyomino a *winner* if the first player has a winning strategy, that is, if he can always force a copy to occur among his pieces. Call it a *drawer* if not, so the second player can force a draw."

"Might it not be possible for the second player to force a win?"

"No. If there was a second-player winning strategy, then the first player could steal it by making a random move, ignoring it, and thereafter following the second-player strategy. If that strategy ever requires him to play where his first piece already is, no

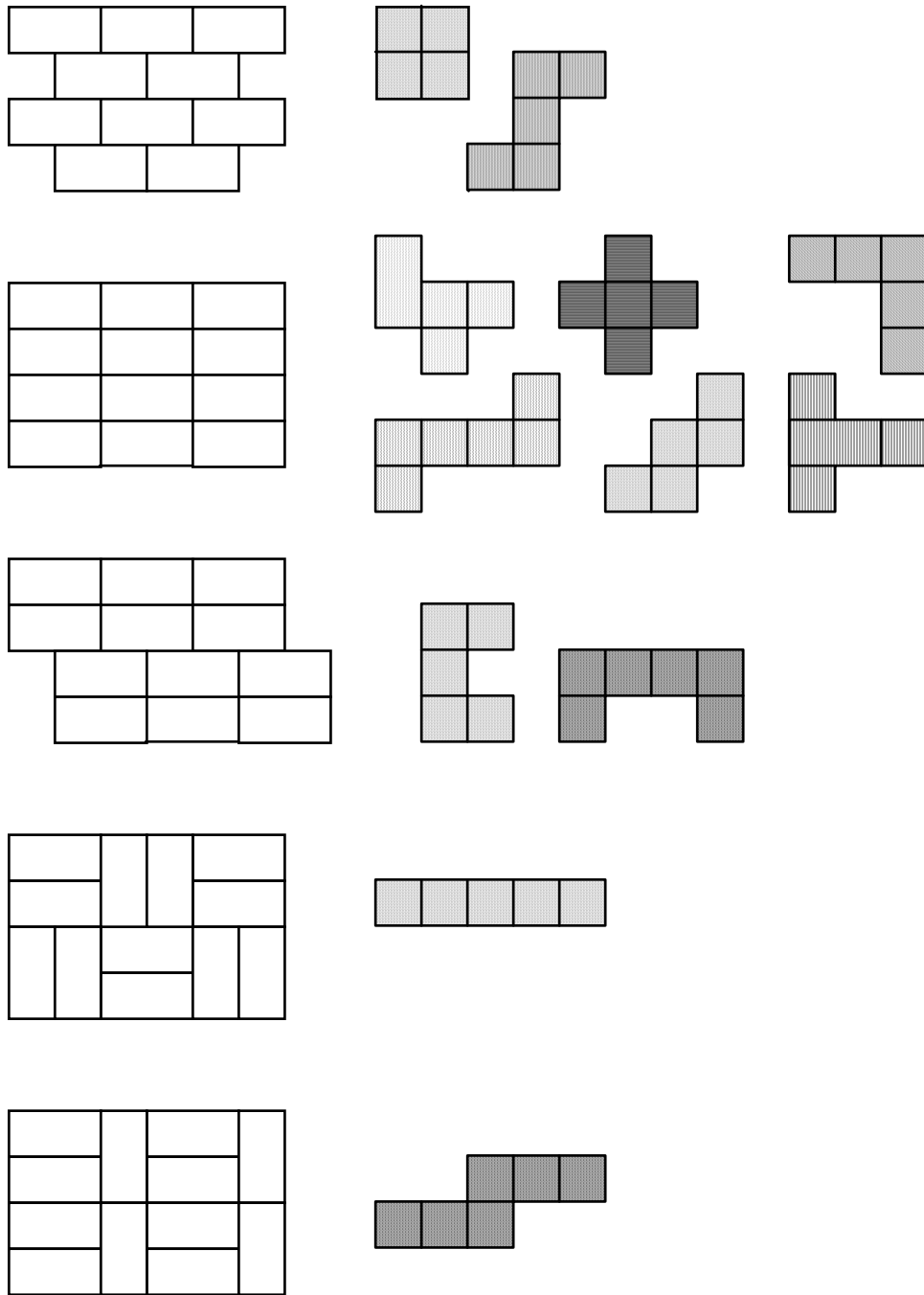
harm is done: he just makes another random move. This 'strategy-stealing principle' shows that a lot of games either have a first-person strategy, or are drawn."

"I've always been worried by that line of argument," said Grimes. "But doesn't it imply the same for chess?"

"No. It only works when the random initial move doesn't alter the subsequent possibilities for the first player. (It can alter the possibilities for the second player — that doesn't matter.) In chess, if you move a piece, you can block other pieces of your own on subsequent moves, so strategy-stealing won't work."

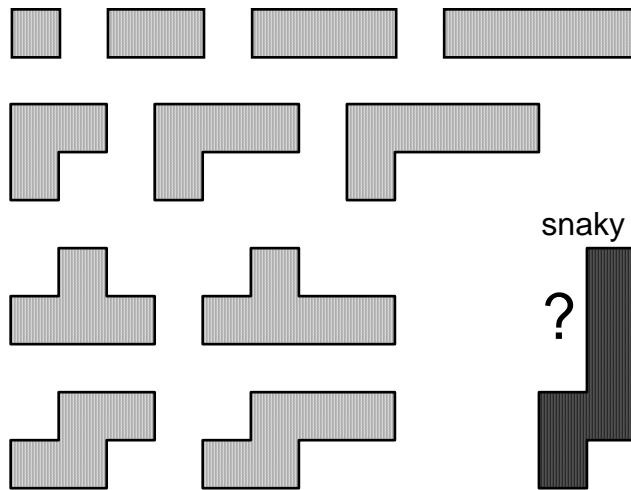
"I see," said Grimes.

"The classification of polyominoes into winners and drawers is *almost* complete," said Bumps. "It's clear that any polyomino that *contains* a drawer must itself be a drawer. Because, if you could force the occurrence of the big polyomino, the same strategy would force the occurrence of the one contained in it — which can't happen since it's a drawer. There are twelve basic drawers, and in every case that property can be proved by a Hales-Jewett domino pairing (**Fig.4**).



Hales-Jewett domino pairings (left) that prove most polyominos (right) are drawers.

Every polyomino that contains one of these twelve is also a drawer. Only twelve polyominos are left (**Fig.5**).



Eleven winners — plus snaky, a probable winner whose status is shaky.

Of these, eleven are known to be winners. The only unsolved case is *snaky*, which is pretty certainly a winner, but to my knowledge there's no absolutely rock-solid proof. (*Readers may enjoy working out winning strategies in each case, and I'd be happy to be told of any proofs for snaky.*)

"Oh, incidentally, you'll notice that one of the pairings excludes the polyomino formed by five squares in a row. So that says that five-in-a-row is a drawn game if diagonal rows aren't allowed — as I told you when we started."

"Those Hales-Jewett pairings are a powerful idea," said Grimes. "I don't suppose they solve any other problems of the same kind?"

"Oh, dozens. For instance, Solomon Golomb found a Hales-Jewett pairing for eight-in-a-row in an  $8_8_8$  cube. Hales and Jewett themselves thought about  $n$ -in-a-row on a  $k$ -dimensional  $n_n\dots_n$  board. They proved that if  $n$  is very large compared to  $k$ , then the game is a draw; but if  $k$  is large compared to  $n$  then the first player can always win. If  $k$  and  $n$  are roughly the same size, nothing is known."

"That's getting rather esoteric, Bumps. How about —" But we shall never know what he was about to say, for at that moment Fumble the sheepdog started barking fit to burst. All the sheep and geese had wandered off while Bumps' and Grimes' attention was distracted. They leaped to their feet, and Fumble, reassured that his master had finally noticed what was happening, lolloped off over the brow of the hill, presumably in pursuit of the vanished livestock.

Bumps looked worriedly at the bare hillside. "I think we've just been wiped out at none-in-a-row."

## FURTHER READING

Elwyn R. Berlekamp, John H. Conway, and Richard K. Guy, *Winning Ways* volume 2: *Games in Particular*, Academic Press, New York 1982.

A.W. Hales and R.I. Jewett, Regularity and positional games, *Transactions of the American Mathematical Society* **106** (1963) 222-229.

Edward Lasker, *Go and Go-Moku*, Dover, New York 1960.