

Murder at Ghastleigh Grange

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It was evening in 221B Baker Street. Coals glowed warmly in the fireplace. Holmes, who had been playing an Irish air, suddenly put down his violin. "If I were you, Watson," he said, "I would turn down the appointment at St. Cuthbert's."

"I was thinking the same myself," I began. Then, in amazement, "Holmes! How did you know St. Cuthbert's had offered me a position, or that I was at this very moment contemplating accepting?"

He put down his pipe. "You know my methods, Watson."

"Please, Holmes. I can't put it in my memoirs unless you explain."

"You have a small mannerism peculiar to the state of deep mental indecision. You reach up as if to tug at your ear, and then draw back. You have talked several times recently of being bored in your current post. Last Thursday you received a telegram. I have heard that the position of chief surgeon at St. Cuthbert's Hospital has become vacant. There, you see how easy it all is."

"Amazing!" His chain of deduction had reminded me of something. "Holmes, did I ever mention that someone has been steaming open all my mail recently? The envelopes have that buckled look to them."

Holmes coughed. "No, Watson, though I admit the same thought had occurred to me. It is a problem with the humidity in our letter-box."

"Ah," I said. "That explains it."

"Speaking of letters," said Holmes, "here is one that has certain features to recommend attention." He passed me a crisp, white envelope. I remarked in passing that it failed to display any signs of buckling in the humid atmosphere of our shared letter-box, but he put this down to the superior quality of the paper.

Inside was a note. There was an ornate embossed gold letterhead,

**Ghastleigh Grange
Grimly Sinister
Hauntingdonshire**

The letter was brief and to the point:

Dear Mr. Holmes,

there has been a terrible murder. Miss Melpomene Beetroot has been bludgeoned to death with a chandelier. The police are baffled. Please help us solve this horrifying crime.

Cornelian, Duke of Ghastleigh (the Third).

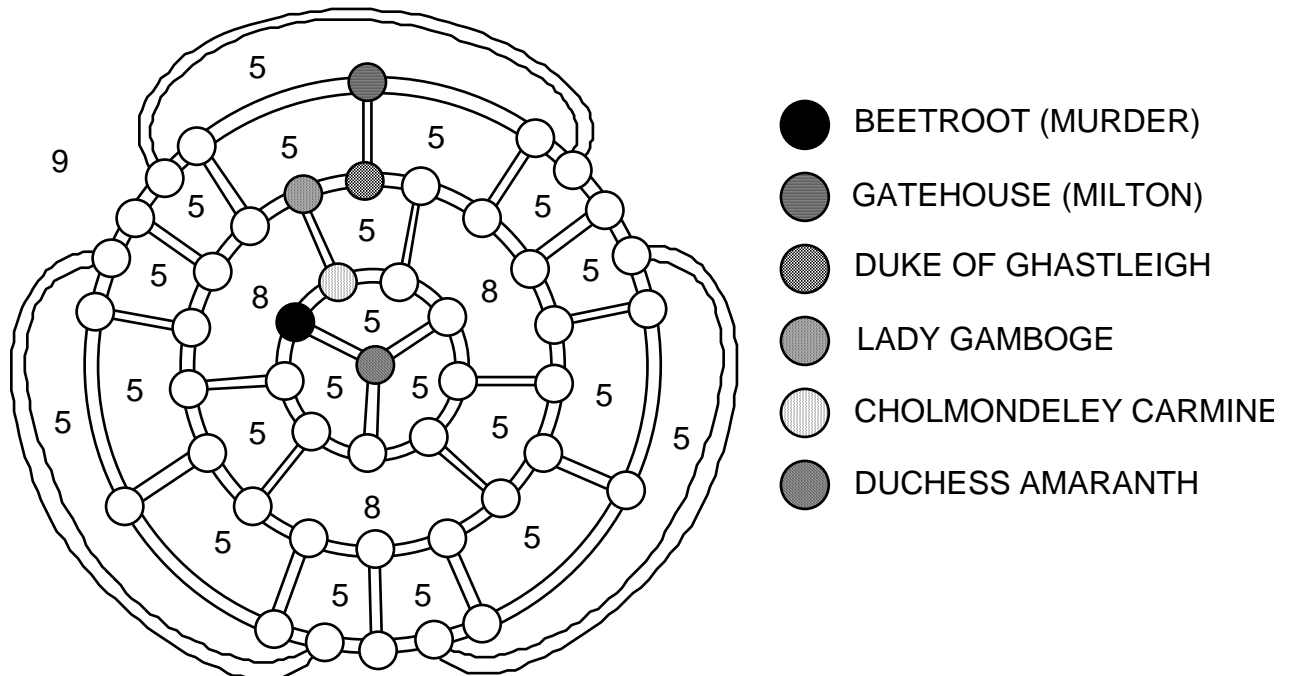
"Miss Beetroot, in the ballroom, with a chandelier?" I hazarded.

"I believe, Watson, that Ghastleigh Grange does not possess a ballroom."

"But where else would one find a chandelier?"

"I have no idea, Watson. Pack our bags for a weekend in the country, and hail a hackney to take us to the station. Such a singular crime will test our wits to their utmost, I do not doubt."

...From the nearby village of Grimly Sinister, just up the lane from Grimly Dexter, Ghastleigh Grange resembled an ancient castle. But as Holmes and I came closer, we realised that it was comprised of some forty-two individual towers, linked together by narrow walkways some considerable distance off the ground. They were arranged in three concentric circles about a single central tower (**Fig.1**).



Plan of Ghastleigh Grange. Numbers show how many walkways surround each region.

The only visible entrance at ground level was on one of the outermost towers, towards which the driveway inexorably led.

We were met at the entrance by the butler, whose name was Dunnett. He led us up a winding stairway and along a walkway to a neighbouring tower, where we were greeted by the third Duke of of Ghastleigh.

"Oh, Mr. Holmes," he wailed. "I cannot thank you enough."

"Have no fear on that account, your lordship," replied Holmes. "My fee will be more than adequate recompense; no gratitude is required. May we inspect Miss Beetroot's boudoir?"

Duke Ghastleigh stiffened. "Good grief, man, how the devil did you know that the murder occurred in her boudoir?"

"If I am not mistaken, your lordship, each tower of Ghastleigh Grange is but a single large room, and each is inhabited by one of the surviving members of the Ghastleigh lineage. Had I said instead of 'boudoir' either 'sitting room' or 'dining room' my statement would have been equally accurate and your startlement equally unjustified."

"This is true," replied the Duke. "Please follow me, gentlemen. We will pass through the towers inhabited by Lady Gamboge and Mr. Cholmondeley Carmine. Dunnett will go ahead and inform them."

"A moment, your lordship. I must send a telegram immediately."

"Dunnett will take care of it, Mr. Holmes. We shall wait, it won't take long." Soon the butler returned and led us to Miss Beetroot's tower.

"This is where the dastardly deed was done," said the Duke. "There used to be a large chandelier hanging from the centre of the roof, but the murderer caused it to become detached in some manner, and poor Melopmene was asleep immediately beneath it. The police have taken away the fragments for analysis."

"Of the chandelier?" I asked.

"Of Melpomene," said the Duke in anguish.

"Who found the body?" asked Holmes.

"I did, sir," replied the butler. "What was left of it, sir. It was a big chandelier and it fell more than fifty feet."

"So you were the last person to have seen Miss Beetroot alive?"

"Apart from the murderer, yes."

I noticed that Holmes, who had been poking about among the wreckage of the room with his magnifying-glass, had stopped, and was listening intently. He rose to his feet, dusting down the knees of his trousers, and replaced the glass in his pocket. "I fear we will find no clues here, Watson," he said. "The police have disturbed the room far too thoroughly, just as they did when I was on the point of convicting that fiend Moriarty in the case of the purloined pterodactyl." He gazed about him. "To whose tower does *this* door lead?"

"It belongs to the Duchess of Amaranth," said Dunnett.

"Might she have killed Miss Beetroot?"

"Dunnett has the only set of keys," said the Duke. "They are extremely intricate of design, and I do not believe it would be possible for anyone to make duplicates. But in any case the Duchess is terribly deaf and spends much of her time asleep. I would not consider her a prime suspect."

Holmes noted this down. "Was everybody in their room on the night of the crime?"

"Indeed, Mr. Holmes," said the Duke. "It is the terms of the family will. The first Duke had a terrible fear of loneliness and he wrote a clause in his will that barred all who did not spend every night at the Grange from inheriting. The terms were binding on all of his successors."

"That's right, sir. Every evening I check all the towers and lock all the communicating doors. I go into each in turn, and lock all the communicating doors; then I move on to another, locking the last door behind me, and so on. In the morning, I make my rounds again in the same manner, and unlock them. Near the end of my round I knocked on Miss Beetroot's door. There was no reply. It was most unlike Miss Beetroot not to be awake, sir, she normally rose with the lark. I became concerned, and unlocked her door. That was when I saw the... remains." His face bore a terrible pallor.

"And nobody entered or left Ghastleigh Grange during the night?"

"No, sir. They never do, sir, and the only way is to pass through neighbouring towers until they reach the entrance tower where I spend every night. I can confirm that nobody passed in or out."

"And everybody was in good health when you locked them up?"

"Yes, sir. They all have bell-pulls in their towers, sir, and once the doors are locked they ring through to the Duke's tower to confirm they are present."

"I use the bells to check that they comply with the terms of the will," said the Duke. "Dunnett is correct. I keep a strict log, and every bell was rung."

"But an intruder could easily ring a bell," I protested.

"No, Dr. Watson," the Duke replied. "Each person in the Grange has a personal code, known only to them and myself."

Holmes turned to Dunnett. "Did you enter any tower more than once, Dunnett?"

"Oh no, Mr. Holmes," cried the butler. "I only enter any tower once when I make my rounds, that's an inviolable rule. It would never do to disturb anybody's privacy once their tower has been locked."

Holmes grunted and tried another line of enquiry. "Your lordship: did the police discover the estimated time of death?"

"It was impossible to tell, Mr. Holmes, because of the state of the body. From the amount that the blood had congealed they thought it was early rather than late — probably before midnight."

Holmes's brow furrowed. "Is there any way that a person could pass between the towers except along the elevated walkways?"

"A skilled mountaineer might perhaps scale the walls from ground level."

"Ah."

"But, Mr. Holmes, he would never be able to do so at night. The Ghastleigh family is very conscious of security. Dunnett's last act on his nightly rounds is to release a pack of hounds into the grounds of the Grange."

"That's right, sir," said Dunnett. "The meanest bunch of Baskervilles this side of the great Grimpen Mire, if you catch my drift, sir."

A boy from the Post Office arrived at the entrance with a telegram for Holmes — no doubt a reply to his earlier missive. He read it, and I saw his eyes narrow. "What route do you take on your rounds, Dunnett?"

"It varies, sir. In truth it is a boring job and I like to choose different routes."

"Can you recall the route you took on the evening before the murder?"

"No, sir."

"That is most unfortunate." Holmes shook his head sadly. "Watson," he said, "as we passed through Grimly Sinister I observed a comfortable village inn. We shall repair to it, and hire rooms for the night. We can do no more here."

"But, Mr. Holmes, the murder —"

"I did not say, your Lordship, that I cannot solve the crime. I was merely remarking that my work *here* is complete. Dr. Watson and I have some work to do, and I have no doubt whatsoever that shortly afterwards I will name the criminal. Dunnett, please call a carriage."

Back at the inn I sank into a comfortable chair. "Holmes, did you really mean what you said to the Duke of Ghastleigh? About being confident of naming the murderer?"

"Watson: since when have I ever dissembled to a Duke?"

"But — there is so little to go on."

"Nonsense, Watson! Why, everything is crystal clear. This telegram confirms it, though its evidence alone is purely circumstantial."

"But who —"

"Patience, Watson, patience. Let us rehearse the pertinent facts. Miss Beetroot was killed before midnight. No intruders could have got in or out because of the hounds. Therefore the murderer was one of the inhabitants of Ghastleigh Grange. Dunnett locked each inhabitant in their room, and they reported their presence to the Duke once he had done so. He started from the entrance, tower, and returned to it, entering each tower precisely once. After the rooms were locked, the only people who could have entered Miss Beetroot's room without being observed were her immediate neighbours. But to do so they must have had a key, and Dunnett possesses the only set of keys. The keys cannot be copied. Who, then, could have committed the murder?"

"Why... nobody," I said, baffled. "Are you suggesting it was suicide?"

"Watson, even the most determined suicide would find it hard to drop a chandelier on their own head. Do you not see who it must be?"

"Uh — oh, of course! Dunnett! He could have returned to her room after she rung her bell to say she was present."

"Precisely. The Duke told us that the Duchess of Amaranth, whose room is adjacent, is as deaf as a post and sleeps heavily. Dunnett could have waited in the Duchess's room until Miss Beetroot rung through, and then re-entered to kill her."

"With a chandelier? How could he have made sure she would wait in the right place? She would have noticed his presence."

"He killed her with another weapon — either a rope or a length of lead pipe, I cannot tell which — and *then* arranged for the chandelier to fall to obliterate the evidence."

"An interesting theory," I told him.

He nodded. "But as yet *only* a theory Watson. How can we *prove* Dunnett was the killer? The Duchess of Amaranth would still be asleep when he re-emerged, and then he would continue his rounds as if nothing had happened."

"The sound of the falling chandelier would surely awaken somebody —"

"The towers are isolated from each other. No, there would be no sounds."

"Dunnett would be delayed on his rounds."

"A few minutes only, if he had already weakened the chandelier's support. Not enough to attract attention."

I struck my forehead a blow with my hand. "Then we are defeated, Holmes! It can be none other than Dunnett, yet the scoundrel must go free."

Holmes laughed. "By no means, Watson. If fortune is with us, he may yet be condemned out of his own mouth." He handed me a sheet of paper. On it was a map of Ghastleigh Grange. "Watson," he said, "here is a simple puzzle for you. Dunnett claims that his evening rounds took him through each tower once and once only. He cannot have moved from one tower to the next except along the walkways. Perhaps you will find such a route for me."

"Certainly, Holmes, though I fail to see what purpose it will serve. There must be *hundreds*. Why, let me just start out like *so*, and... Oh. Well, in that case — oh."

An hour later I pushed the paper aside in a fit of pique. "Well, there must be a solution, Holmes, but I'm damn'd if I can find it."

"In truth, Watson, I suspect that there is not. I have asked you to find a *Hamiltonian circuit*: a closed loop through a network that visits each node precisely once. It is named for Sir William Rowan Hamilton, who marketed a puzzle asking for just such a circuit along the edges of an icosahedron. No foolproof method is known, beyond systematic trial and error, that can determine whether a given network has a Hamiltonian circuit."

"Then the butler has cheated the noose, Holmes, for the network is far too large to be analysed by trial and error. Miss Beetroot will never be avenged."

"Not necessarily, Watson. I hope that in this case we may be fortunate. Two Russian mathematicians, by name of Kozyrev and Grinberg, have devised a condition that must be satisfied by any network in the plane that possesses a Hamiltonian circuit. We will see whether it is satisfied by the towers of Ghastleigh Grange." He took the paper from me. "But you must check my working."

"Suppose the network has n nodes. Imagine drawing such a circuit, Watson, on the network (**Fig.2**).

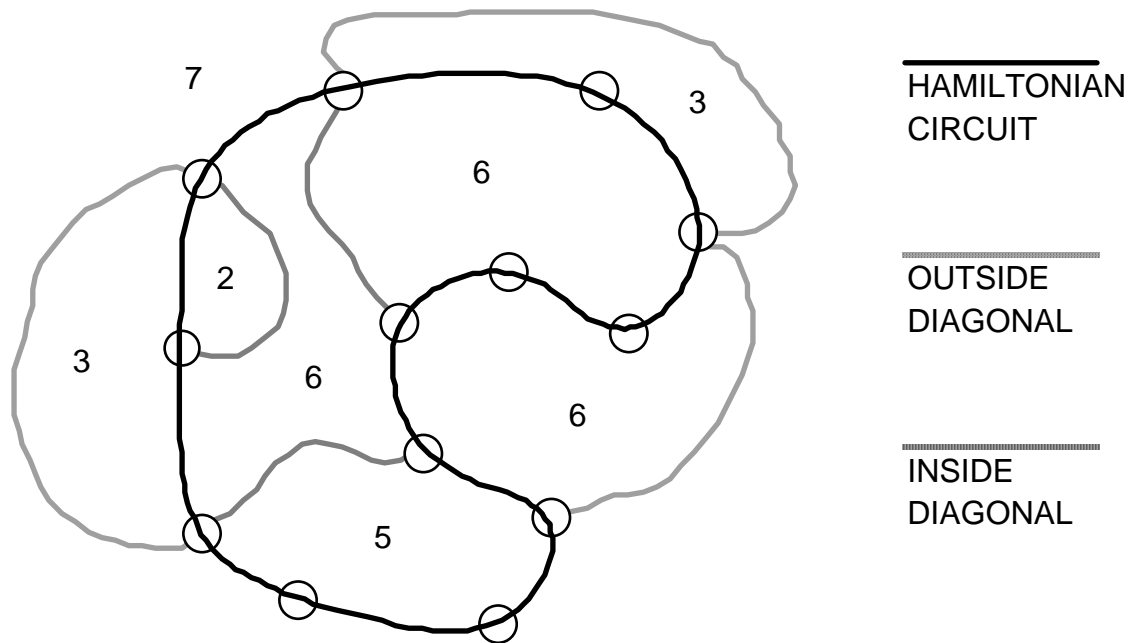


Illustration of the Kozyrev-Grinberg formula. This network has $n = 13$ nodes, connected by a Hamiltonian circuit (solid line). There are $d = 3$ inside diagonals creating $d+1 = 4$ regions with 2, 5, 6, and 6 sides respectively. Therefore $f_2 = 1$, $f_5 = 1$, $f_6 = 2$ and all other f_j are zero. Similarly there are four outside regions with 3, 3, 6, and 7 sides, so that $g_3 = 2$, $g_6 = 1$, and $g_7 = 1$. The Kozyrev-Grinberg formula says that $(0-2)+3(1-0)+4(2-1)+5(0-1) = 0$, which is correct. Similarly each step in the derivation of the formula can be checked for this example.

Then it has n edges, since it visits each node precisely once. Those edges that do not belong to the circuit run 'diagonally', either across the interior or the exterior of that circuit. The interior is divided into a certain number of regions by those diagonals that cross it.

"If there are d diagonals, this number must equal $d+1$. To see why, imagine adding the diagonals one at a time. There is one region to begin with, and each diagonal produces a further region. Now, there is an alternative way to count the number of regions. Each region has a certain number of sides — edges of the network that surround it. Suppose that for each integer j , precisely f_j of these regions have j sides. Then the total number of regions in the interior is also given by $f_2+f_3+\dots+f_n$. So $f_2+f_3+\dots+f_n = d+1$. Of course," he added, "many of these f_j will be zero.

"Next, I count the number of edges that surround these regions in two different ways. Any region with j sides is bounded by j edges, so such regions contribute jf_j to the total. At first sight the total would appear to be $2f_2+3f_3+\dots+nf_n$. However, in such a count each of the d diagonals is counted twice, once for each of the two regions that it

abuts; but the n edges of the circuit are counted once only. Therefore $2f_2+3f_3 +\dots+nf_n = 2d+n$.

"Doubling the first equation and subtracting it from the second, I find that $f_3+2f_4+3f_5 +\dots+(n-2)f_n = n-2$. There is a similar equation for the exterior of the circuit, $g_3+2g_4+3g_5 +\dots+(n-2)g_n = n-2$, where g_j is the number of regions *outside* the circuit that have j sides. Finally, subtracting one equation from the other, I attain my goal, the *Kozyrev-Grinberg formula*:

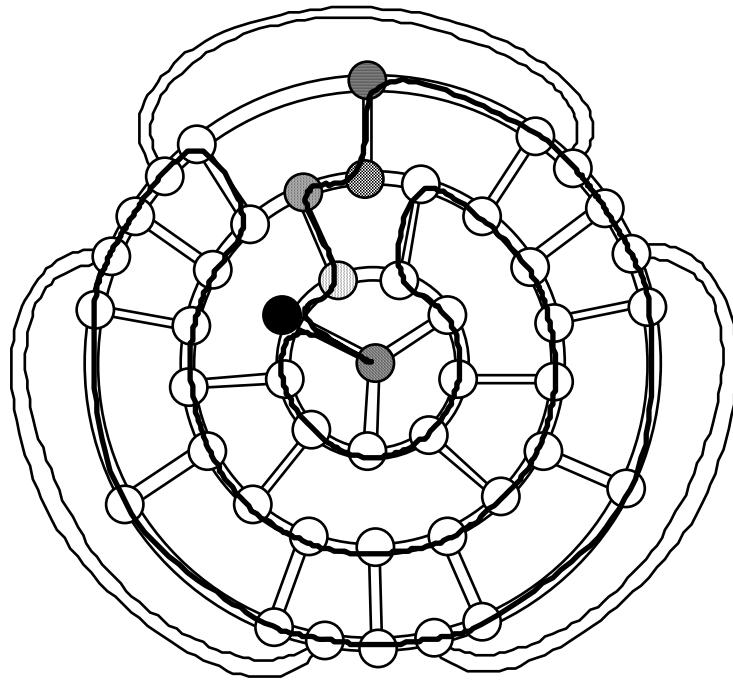
$$(f_3-g_3)+2(f_4-g_4)+3(f_5-g_5)+\dots+(n-2)(f_n-g_n) = 0."$$

"Elegant, Holmes, and assuredly subtle; but I fail to see the relevance to Dunnett's guilt. We have no idea what the values of f_j and g_j should be. Indeed, if there is no Hamiltonian circuit, there are no such values at all."

"I am hoping that the assumption of a circuit will lead to a logical contradiction, Watson. I have told you before that once you have eliminated the impossible, then whatever remains, however improbable, must be the truth. Now, Watson, if you examine the network for Ghastleigh Grange you will find that all of its regions have either 5, 8, or 9 sides. If it has a Hamiltonian circuit, then by the Kozyrev-Grinberg formula $3(f_5-g_5)+6(f_8-g_8)+7(f_9-g_9) = 0$. However, there is only one 9-sided region, the entire outside of the network, so that f_9-g_9 is either 1 or -1. Yes, we have the wretch. For now we are logically forced to conclude that $\pm 7 = 3(f_5-g_5)+6(f_8-g_8)$, manifestly a multiple of 3, which is impossible."

"Therefore *no Hamiltonian circuit exists!* Dunnett is lying. Holmes, I am speechless with admiration."

He smiled at the compliment. "Would I were so fortunate, Watson. Dunnett *must* have visited at least one tower twice. The only reason why he should lie is that it was Miss Beetroot's tower to which he returned. And indeed there is at least one possible route that visits all the towers — except Miss Beetroot's — once only; and it re-enters from the Duchess's tower (**Fig.3**).



A possible route for Dunnett.

All the evidence is in place, Watson. Tomorrow morning we shall return to Ghastleigh Grange and confront Dunnett with the evidence of his mendacity. I have no doubt that the poor devil will break down and confess, so it would be wise to have the police waiting to take his statement."

"Brilliant, Holmes. But what put you on to him?"

"The Duke failed to mention Dunnett's first name, and I became suspicious. In order not to arouse the butler's suspicions by enquiring openly, or risking being overheard when asking the Duke, I dispatched a discreet telegram to Scotland Yard to check their files."

"And you found that Dunnett had a criminal record?"

"Nothing as simple, Watson. I found that his name was Hugh."

I scratched my head, baffled. "How did that help, Holmes?"

"You know my methods, Watson.. Apply them. What would *you* have concluded had you been investigating a murder and received a telegram from Scotland Yard telling you 'Hugh Dunnett'?"

FURTHER READING

Russ Honsberger, *Mathematical Gems I*, Dolciani Mathematical Expositions 1, Mathematical Association of America 1973.