

Nontrivial Pursuit

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The *retiarius* leaned forward like an avenging demon, his trident poised for the final lunge. His victim, hopelessly entangled in a heavy net, looked up with imploring eyes. The crowd packing the benches of the vast circular amphitheatre roared in a frenzy of bloodlust. All eyes turned to Caesar's box... the Emperor raised his hand — thumb down. The gladiator hurled his trident, missing his victim from point-blank range, and tangled it in the net. Stepping forward to retrieve the weapon, his foot too became ensnared. Soon the two gladiators, both weaponless, were inextricably entwined in a futile struggling mass by the twisted ropes of the net.

The Emperor Scandalus turned to his companion, the lady Dyspepsia. " 'Tis not quite the spectacle I had hoped, my dear."

"Nay. 'Tis *almost* as enthralling as watching duckweed grow, but, given the choice, my preference would be for duckweed."

Scandalus signalled to Fastidius Finici, one of his proconsuls. "Inform Barnumus Bailius that unless his circus puts on a better show, I will have him thrown to the — of course! That's it!"

"What is it, O Great Caesar?"

"Lions! Why don't we pit gladiators against lions? That will ensure that the lazy beasts put some effort into killing each other."

"The lions, O Caesar?"

"No, you fool, the gladiators! Nothing like a huge hairy lion breathing down your neck to get the adrenalin flowing. Tell Barnumus to get himself a few lions in time for next week's contest."

Fastidius hastened to obey. "Lions," said Barnumus, rolling the word round his tongue like an exotic sweetmeat. "Well now. Lions." He leaned confidentially towards Fastidius. "Now your actual *lions*, Fastidius, they aren't too easy to come by in these decadent times. My aunt Impedimenta has a pretty fierce *cat*, though...No, wait, I can see that you're not keen on a cat. A sheep? A goat? A cow with really terrific 'orns, you wouldn't believe — no, I see it's a lion or nothing. Well, since it's for you, I might *possibly* be able to lay hands on one by the Ides of October..."

Knobblines, the Emperor's Greek geometry tutor, spoke up. "Caesar wants the lions for next week's gladiatorial contest, Barnumus. He's not prepared to wait till October."

"Oh, it's that fool geometer." He spat on the sandy floor. "Totally useless, geometry is, to a circus man like me. So push off, protractorface." He turned back to

address Fastidius. "I could do you a real bargain if you're not in a hurry, friend of mine knows where to get a couple of dozen lions that fell off the back of a chariot... now, there's no need to shout. If the Emperor Scandalus wants lions, lions he will get."

"Good. And make sure they are pitted against your best and most audacious gladiators."

"I will start the combat with the gladiator Egocentrix, single-handed against the most ferocious of my lions, armed only with a short sword."

"Egocentrix?"

"A Gaul from the tribe of the Tricidici."

"Excellent, nothing sets the game up better than an early Gaul. But I do have one piece of advice for you, Barnumus."

"Yes?"

"You don't need to arm the lion with a short sword. It can take care of itself without."

A week passed.

"Now *he* looks a fine specimen," said Dyspepsia. "Look at those muscles! Who is he to fight?"

"You might better ask 'what'."

"I take not your meaning."

"I little surprise, my dear. Look!"

Lady Dyspepsia gasped. "A lion! A fierce, wild, untamed — Scandalus, why is it rolling on its back with its paws in the air?"

"To show its ferocity, my dear."

"Then why is that boy tickling its stomach?"

"He is — er — a leotitillarius. He is there to — um — goad it into a frenzy. 'Tis a dangerous task."

"Oh." She leaned forward, peering myopically. "The poor beast looks terribly mangy, Scandalus."

"Nonsense! One of the finest purebred lions from the plains of Africa."

Egocentrix began to approach the lion, his sword held at the ready. The lion yawned, curled itself up in a ball, and went to sleep. Egocentrix took courage from this, and tentatively poked his sword at the animal's haunches. The lion, enraged by being so unkindly awoken, leaped to its feet and roared. Egocentrix yelped, dropped the sword, and began to run; the lion followed in hot pursuit. After ten minutes rushing round and

round the arena Egocentrix realised that the lion wasn't very quick on its feet, and slowed down to a trot. Whenever the lion approached too close, he sprinted away again. Soon the lion become bored, forgot what the source of annoyance had been, and went to sleep again.

"...but Great Caesar, it is impossible to procure a wild lion at such short notice!" protested Barnumus. "I was forced to borrow one from Squintus Dubius, who keeps them as pets. In any case, the animal became fierce enough when jabbed from behind with a sword."

In order to interview Barnumus, the Emperor had interrupted his geometry lessons from Knobblines, which he greatly enjoyed, and so was in a poor mood. "The mistake, Barnumus, was not so much in the lion as in the gladiator. There is no spectacle in a contest when the man can outrun the lion."

"Great Caesar means," put in Fastidius, "that if the man's speed is greater than the lion's, then the man can always escape from the — ha ha — paws of death."

"Indeed, O Caesar, there is a simple geometric proof of that theorem," said Knobblines. "Except, of course, if one subscribes to the paradoxical views of Zeno, who purports to prove that all motion is impossible."

"With which your lion seemed to concur," said Scandalus.

"I will replace Egocentrix by Geriatrix, O Caesar. He can scarcely stumble a dozen paces."

"There is also no spectacle if the lion can outrun the gladiator," put in Knobblines.

"You keep your Grecian nose out of this," threatened Barnumus.

"No, he is right. Just a few quick bites and it is all over," said Caesar glumly. "Barnumus, you must find a lion and a gladiator that are perfectly matched, each able to run at the identical speed, so that neither has any obvious advantage. Then we will see a contest indeed."

"I would not be so sure of that, O Caesar," said Knobblines.

"By the gawds, he's off again," said Barnumus.

"I am wondering what happens if the gladiator emulates what Egocentrix did today, and runs as fast as he can away from the lion."

"Then the lion just runs after him," said Fastidius.

"Indeed. And at the same speed, so that while the gap between them does not increase, neither does it diminish. Not, to my mind, an especially enthralling tactical battle."

"It would be a little boring," Caesar conceded. "But the man cannot run away indefinitely, or he will encounter the wall of the arena."

"An excellent point, Caesar," said Knobblines. "It is an interesting question. If a lion should pursue a gladiator in a circular arena, and if each runs at the same (constant) speed, can the lion always catch the man? Or can the man, by cunning changes of direction, keep the lion at bay indefinitely?"

"It seems to me," mused Caesar, "that whenever the gladiator turns through an angle, the lion can take a short cut and close the gap between them a little. Moreover, the gladiator cannot keep running straight, or he will encounter the wall."

"True," said Knobblines, "but if the gladiator follows a smoothly turning path, then the change in direction at any given instant is vanishingly small."

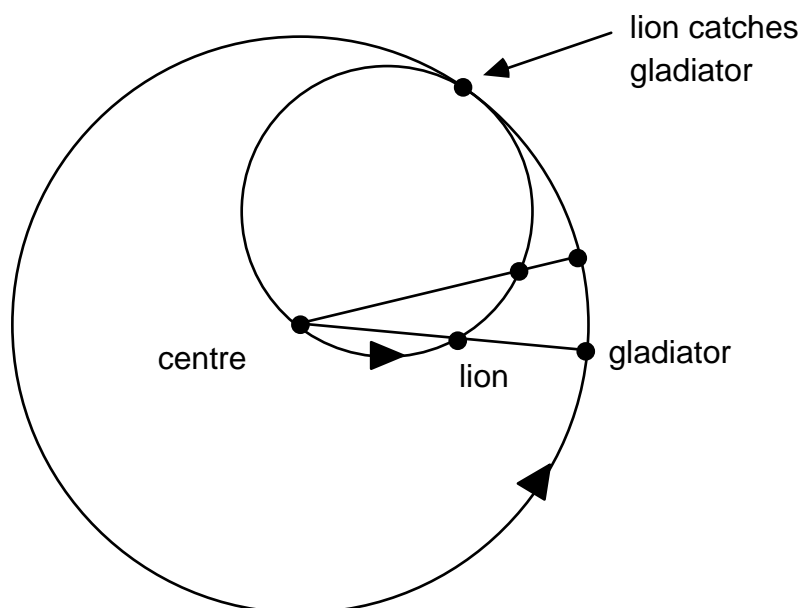
"Well," said the Emperor in some irritation, "it just seems to me that if the lion stays between the man and the centre of the arena — "

"On the radius joining the man to the centre of the circle?" asked Knobblines.

"Precisely. If the lion does that, then it can — in a manner of speaking — push the man out towards the wall, and trap him there. It has a smaller distance to move when it goes *around* the circle, and that leaves some spare movement along the radius. Because the lion is on the inside of the man."

"And soon the man is on the inside of the lion," laughed Fastidius.

"An excellent suggestion, O Caesar. However, I do wonder whether perhaps — as Zeno hypothesized for Achilles in pursuit of the tortoise — the lion may approach indefinitely close to the man without ever actually reaching him. The advantage of extra radial speed diminishes as the lion approaches the man. Hmm... Suppose for simplicity that the man runs in a circle. Then the lion can run along a smaller circle that is tangent to it (**Fig.1a**).

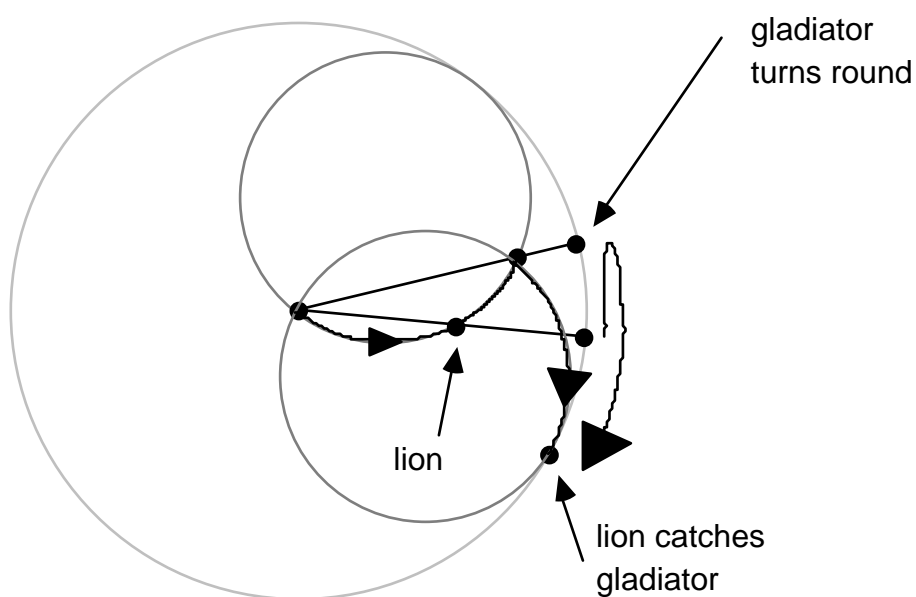


(a) How a lion can catch a gladiator who moves in a circle at constant speed.

Elementary geometry proves that corresponding arcs of the two circles possess identical lengths, so such a track is possible. In which case the lion does indeed catch the man after a finite period of time."

"Elementary geometry," muttered Barnumus darkly. "Your actual gladiator ain't so stupid as to keep running in the same direction when he sees a flaming *lion* closing in on him, is he? No, he'll turn on his heels and run the other way!"

"But the the lion can simply reflect its circular path in that radial line, and continue exactly as before," Knobblines cried in triumph (**Fig.1b**).



(b) How the lion should modify its tactics if the gladiator reverses direction.

"Barnumus, you must procure a lion whose speed precisely matches that of your most skilful gladiator, and train it to stay on the same radius as the gladiator while it chases him. That way, the circus crowds will be treated to an exciting spectacle as the gladiator rushes frantically to and fro trying to elude the ravenous beast. Eventually, the two must fight."

"Cor, don't want *much*, does he?" Barnumus muttered under his breath, but not so loudly that Caesar might hear. Otherwise, Barnumus realised, he might well be seeking a lion that could run exactly as fast as *him*.

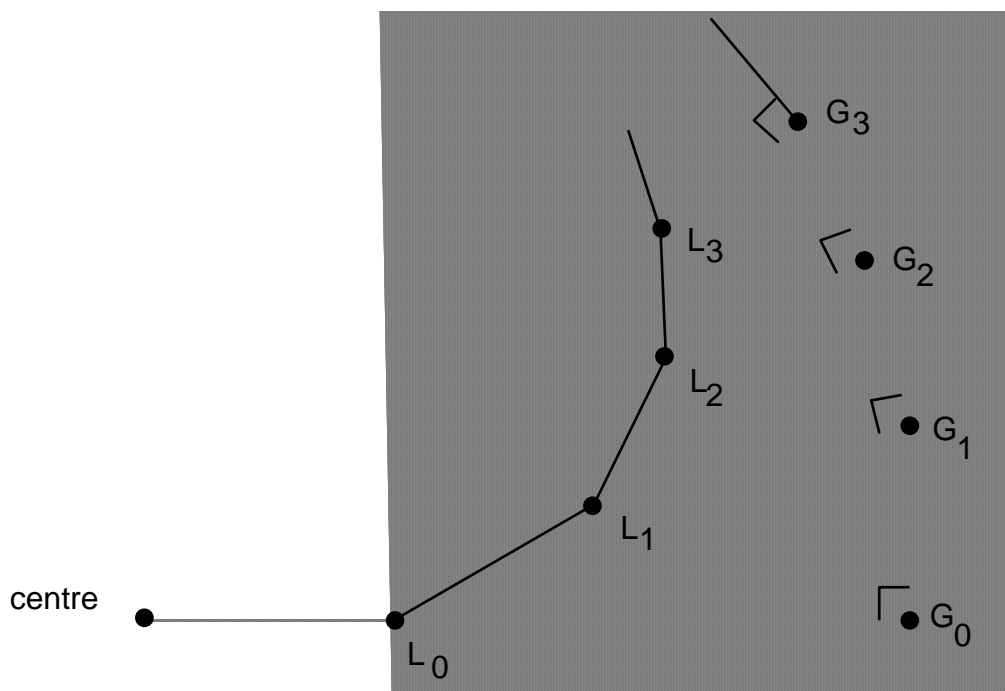
Another week passed, while Barnumus conducted speed trials on lions and gladiators, and embarked on a crash training course involving large quantities of fresh gazelle as a reward for radially synchronised prowling.

The noise was deafening as the lion's cage was conveyed to the centre of the arena on a kind of crane constructed by Knobblines, and released by a cunning arrangement of levers. The lion headed straight towards Egocentrix the Gaul, who let out a screech and rushed off at right angles to the lion's path. As it had been trained to do, the lion quickly turned, to stay on the same radius. Noticing this, Egocentrix once more turned at right angles to the radial line. The lion steadily closed the gap, but at a slower and slower rate. It didn't look at all like Knobblines' diagram of two circles.

"Whatever is the fool doing?" asked Scandalus.

"I have no idea, O Caesar," said Fastidius. "But the lion is pursuing his Gaul with singleminded attention."

Knobblines slapped himself on the thigh. "Of course. Egocentrix is following a *squiral*. A spiral made up from successive straight line segments, each at right angles to the radius (**Fig.2**).



Constructing a squiral. Diagram shows successive positions of gladiator (G_n) and lion (L_n). Each segment $G_{n-1}G_n$ is at right angles to the corresponding radius, and has length proportional to $n^{-3/4}$. The total length is infinite, the path stays within the circle, and if the lion stays on the radius it can never catch the gladiator.

Simple geometric arguments now demonstrate that, if it stays on the radius joining Egocentrix to the centre of the arena, the lion cannot catch him while he is running along any particular segment. And therefore it cannot catch him at all — even though it may approach indefinitely close."

"But if he — er — squirals ever outwards, he must run into the wall eventually," objected Fastidius.

"Not necessarily," said Knobblines. "If the radial component of the motion covers shorter and shorter distances, then the wall may never be reached. I have in mind

a series such as $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$ which never exceeds 2, no matter how many terms are added."

"But surely," said Scandalus, "in that case his motion around the circumference of the arena will *also* remain under some limit, so the lion will soon catch up with him."

"I fear not," said Knobblines. "If the length of the n th segment of his path is proportional to, say, the fourth root of the cube of the reciprocal of n , then a short calculation shows that the radial distance remains bounded but the angular distance can be made to exceed any given quantity. I regret that Egocentrix can continue his squiral indefinitely without ever reaching the wall, or being caught by the lion."

"Have Barnumus brought to me," said the Emperor grimly.

"...Knobblines, me old mate, you've got to help me!" wailed Barnumus. "Now I've got to *untrain* the lion, so that it's free to follow its own inclinations, *and* I'm going to be the one what's doing all the running!"

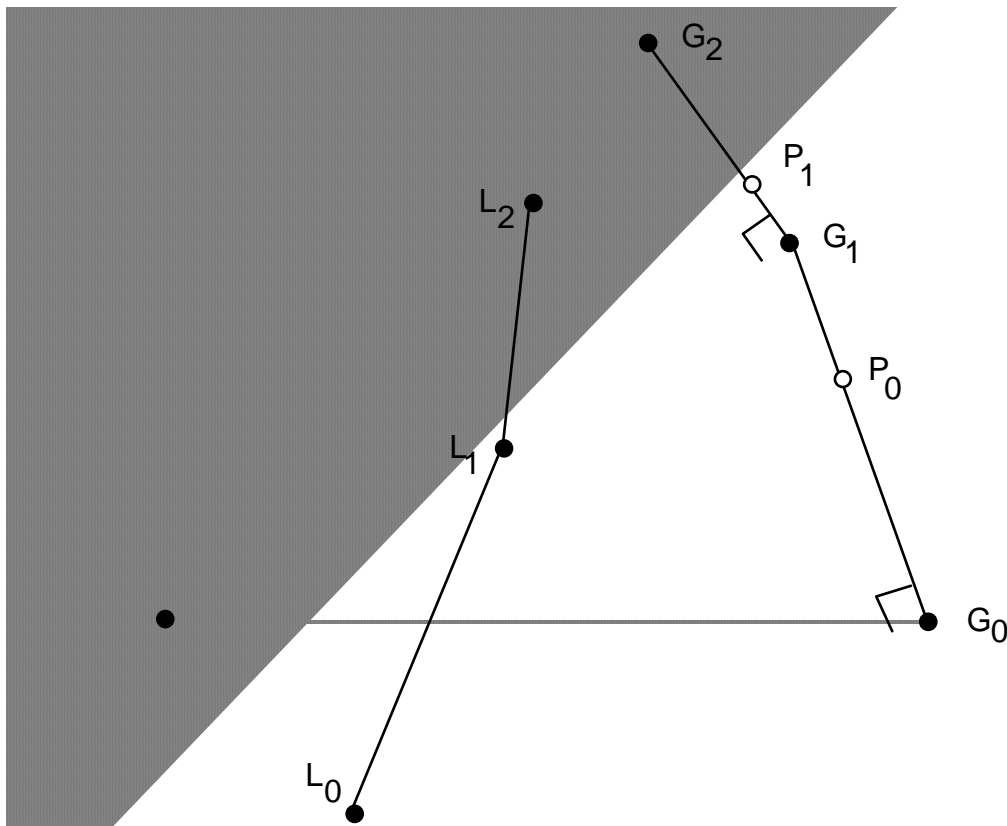
"Bad luck you run at exactly the same speed as the lion," Knobblines said, in apparent sympathy.

"Will of the gods. I knew I ought to have paid more dues to the vestal virgins... *please*, Knobblines."

"Well... you can't follow a squiral any more, because the lion can cut across the corners... But maybe we can do something about that..."

If anything, the amphitheatre was more crowded than usual. Barnumus was not the most popular person in Rome. "I think I'm going to enjoy this," said Scandalus.

"Barnumus Bailius will provide the lion's share of our entertainment, I suspect," said Fastidius with a grin. The beast was released: like Egocentrix before him, Barnumus set off at right angles to the line joining him to the lion. The lion cut across towards him, abandoning its failed radial strategy and heading roughly towards where it thought Barnumus was going. Barnumus ignored it, but suddenly changed direction, again at right angles to the line between himself and the lion (**Fig.3**).



Modifying a squiral to avoid capture no matter what the lion does. Assume the lion starts at L_0 , the gladiator at G_0 . The gladiator moves perpendicular to L_0G_0 , through P_0 on the parallel radius, and continues for a distance P_0G_1 past it that is equal to the first segment of the squiral. By that time the lion has moved, say to L_1 . Now repeat the construction, so that P_1G_2 is equal to the second side of the squiral, and so on. Again the total length is infinite, the path stays within the circle, but now whatever the lion does it can never catch the gladiator.

"He won't keep this up much longer," said Scandalus, rubbing his hands in glee.

"I hope not, O Caesar," Dyspepsia whined in complaint. "This running is so tiresome. When will we see some good red blood?"

"Egocentrix will stop and fight when he realises that it is inevitable that the lion will catch him," said Caesar. Knobblines nodded in apparent agreement, but his thoughts belied his expression. *You'll have a long wait.* He reviewed the strategy in his mind, testing the logic again. First, construct a squiral according to Egocentrix's original method, assuming the lion keeps to the radius. But then modify the path according to the lion's actual position. Begin by moving at right angles to the line joining lion and man. Continue in that direction until encountering the radius parallel to that line, and extend the path beyond it by a length equal to the first segment of the squiral. At that point, observe where the lion has moved to, and repeat the construction, once more moving at right angles to the line between man and lion, across the parallel radius, and beyond it by

a distance equal to the second segment of the squirrel. And so on. Again, elementary geometry proved that, along each such segment of Barnumus's path, he could not be caught by the lion; that the path remained always within the arena's walls; and that its total length exceeded any finite bound...

Afterwards, an out-of-breath Barnumus walked over to Knobblines and shook him by the hand. "Brilliant idea, mate. Worked like a charm."

"Yes. Pity the lion got fed up and went to sleep."

"Better than getting fed *me* and going to sleep."

Fastidius appeared. "Ah, the star of our show. Barnumus, I have welcome news for you."

"Oh, good."

"So wonderful was your performance that Caesar has decided to let you repeat it next week."

"I'd really prefer to retire while my career is at its peak — "

"Against *two* lions, this time." Fastidius turned on his heels. Barnumus looked imploringly at the Greek.

"Against two lions, you're a dead duck, I'm afraid," said Knobblines. "Unless Caesar rebuilds the arena, expanding it into the third dimension, thereby making it a sphere. It can be proved that n lions all running at the same speed as a man can always catch him in an n -dimensional ball; but $n-1$ cannot if he adopts the right tactics. Perhaps I can persuade the Emperor that your demise will be too swift to be interesting. Then I might also persuade him to have some obstacles placed in the arena for you to hide behind."

"Will that keep me safe?" asked Barnumus.

"The point is moot. One lion cannot catch you if there is an obstacle: you merely manoeuvre until it is between yourself and the beast, and keep it there. But it is an unsolved question whether two lions can always be avoided when there are several obstacles. You must exert every effort to find the solution, Barnumus. If the goddess of Fortune smiles, you may yet survive."

"But I don't know any geometry!" wailed the circus-owner.

"You should have realised it would come in useful," said Knobblines.

"Why should a circus man imagine something as ridiculous as that?"

"Because everybody knows that ruling lots of lions is what geometry is all about," said Knobblines.

FURTHER READING

Béla Bollobás (editor), *Littlewood's Miscellany*, Cambridge University Press, Cambridge 1986.

Hallard T. Croft, "Lion and Man", a postscript, *Journal of the London Mathematical Society* **39** (1964) 385-390.