Pyramid Power

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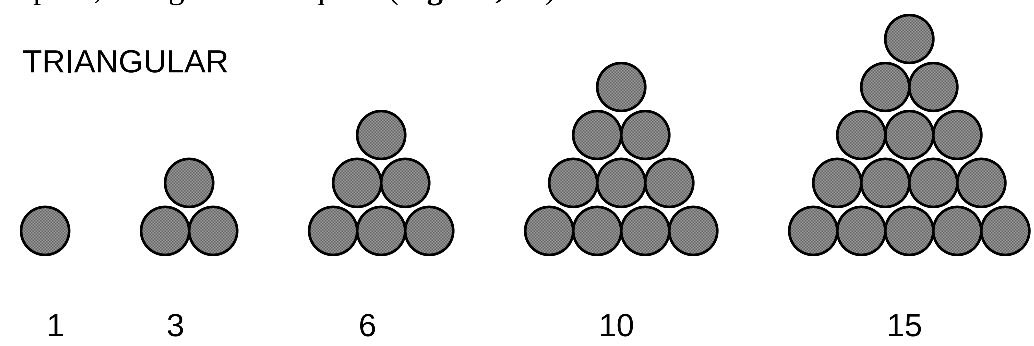
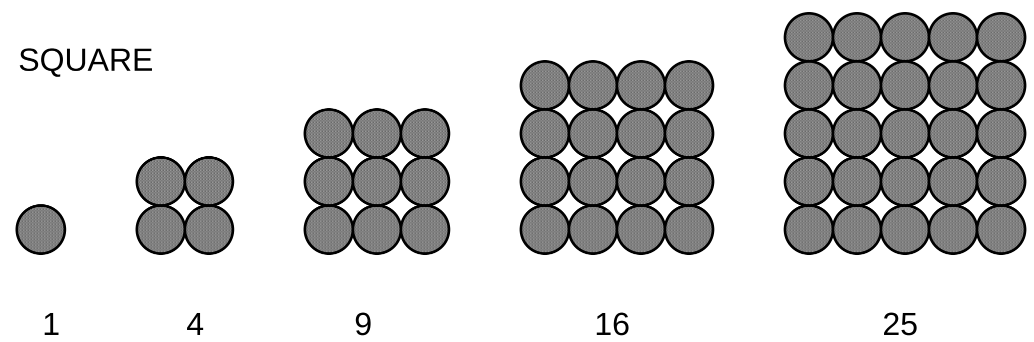
The Spherical Sportswear Shipping Shop was changing its corporate image. Billy Jo Rottweiler had been promoted from Tennis Ball Packer Grade 3 to Product Enhancement Executive — leading to a salary *cut* that she had been assured was a necessary feature of the new grading system — and her assistant Silas Golding was now an End-use Assembly Operative.

It had all come about because SSSS had brought in a firm of outside efficiency consultants, called Market Force. ('Market Farce', more like, Silas had said, and events so far had proved him right.) The first thing they did was change the names of all the jobs so that nobody knew any longer what anybody did, themselves included. It was rumoured that they were about to make a seriously radical restructuring by redesigning the company's headed notepaper. However, one thing they definitely were doing was changing SSSS's packaging. Billy Jo had been permitted to attend a meeting about it, as a non-voting observer.

"Agenda item 1: packaging," said Hugh Jego, the Consumer Preferences Advisor from Market Force. "You'll see that I have prepared a list of all the types of packaging currently used by SSSS. Mainly you use cardboard cylinders, which in the consultancy business we generally refer to as the in-line-unipack system — terribly dated, that. These units are often wrapped in multi-pack assemblies. In addition you have two types of flat pack, triangular and square (**Figs 1a, 1b**)."



(a) Triangular numbers.



(b) Squares.

"Yes," said Orwell O.Osborne, SSSS's Near-Market Product Development Team Leader, whatever that meant. "We introduced the triangular packs for spare sets of red snooker balls, because fifteen red balls form a triangle. That's how they're arranged at the start of the game. And the square packs are suitable for bowls, where each player normally uses a set of four woods."

"But you produce packs containing many other numbers."

"Yes, we like to have a complete range available."

Jego consulted his list. "Including square packs up to 200_200. Who could possibly need *forty thousand* balls?"

"We introduced that size for Grand Slam tennis championships," said Osborne. "They use a lot of tennis balls and we were hoping to attract the bulk-buy market."

"How many such packs did you sell?"

"Um — let me look it up and — uh — none."

"And you also sold none of the 199_199, 198_198, and so on right down to 7_7."

"No, but we don't like to have a gap in the range —"

"The entire *range* is a gap, Mr. Osborne. It represents a huge waste of resources. Why, the warehousing requirement alone eats up substantial sums." Jego waved his list in dismissal. "In any case, flat packs are totally old-fashioned."

Osborne leaned his chin on his hands. "So what do you propose?"

"Hyperspatial packaging," said Jego. "The great leap from two dimensions into three, first made by the winged insects of the mesozoic! A world unexplored by humankind until the days of the Wright Brothers!"

"My middle name is 'Orville'," said Osborne. "Orwell Orville Osborne."

"We call him 'triple zero'," said Billy Jo in a stage whisper, to nobody in particular.

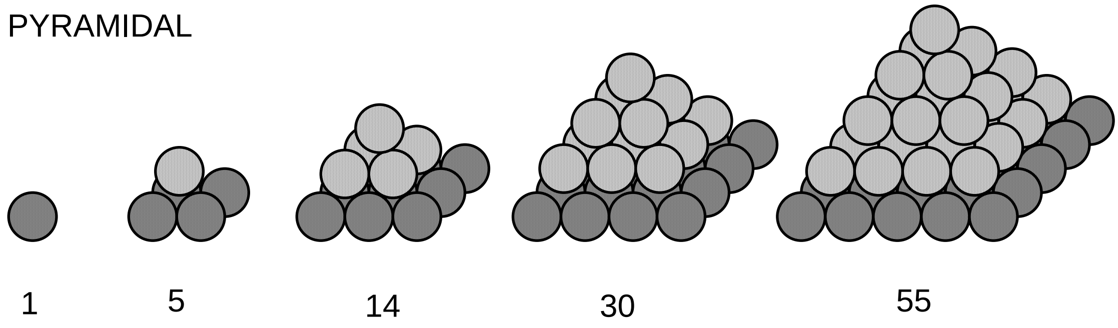
"And from three dimensions into four, five, a thousand, a googolplex!" Jego went on, oblivious to the byplay.

"How do you package tennis balls in four dimensions?" asked Billy Jo.

"First we must explore the possibilities in three dimensions," said Jego. "Then I will explain how to enhance the dimensionality of the product."

"Oh."

"One favoured geometry for three-dimensional packaging is the square-based pyramid," said Jego. "As you all know, the pyramid is a powerful symbol going back to ancient Egypt, and what's more it stops razorblades from going rusty." Billy Jo shook her head sadly. "You can make pyramidal packs by stacking different sizes of square pack on top of each other (**Fig.2**), thereby using up your existing flat-pack stocks."



Pyramidal numbers, obtained by stacking consecutive squares.

But it wouldn't be that easy, Osborne pointed out. The warehouse held different numbers of square packs of different sizes, and after a while some sizes would run out.

"Couldn't we break up the packs and reassemble them?"

"Not if we break open too many at once — it'll take ages because everybody will keep getting confused. I recommend tackling one pack at a time."

"Only *one* pack..." mused Billy Jo. "Hey, I've noticed a strange coincidence. I've been working out the numbers of balls in big square pyramids. I call them *pyramidal numbers*, just like we call the numbers for square packs *squares* and those for triangular packs *triangular numbers*. So the pyramidal numbers P_n begin

$$P_1 = 1$$

$$P_2 = 1 + 4 = 5$$

$$P_3 = 1 + 4 + 9 = 14$$

and so on. Now, calculating the bigger pyramidal numbers takes quite a long time, but I've noticed a short cut. Just as there is a simple formula for the n^{th} square —"

"Such as?" asked Osborne.

"Well, n^2 of course. And the nth triangular number is n(n+1)/2. You might use the symbols

¶$_n$= n^2

Î$_n$ = n(n+1)/2.

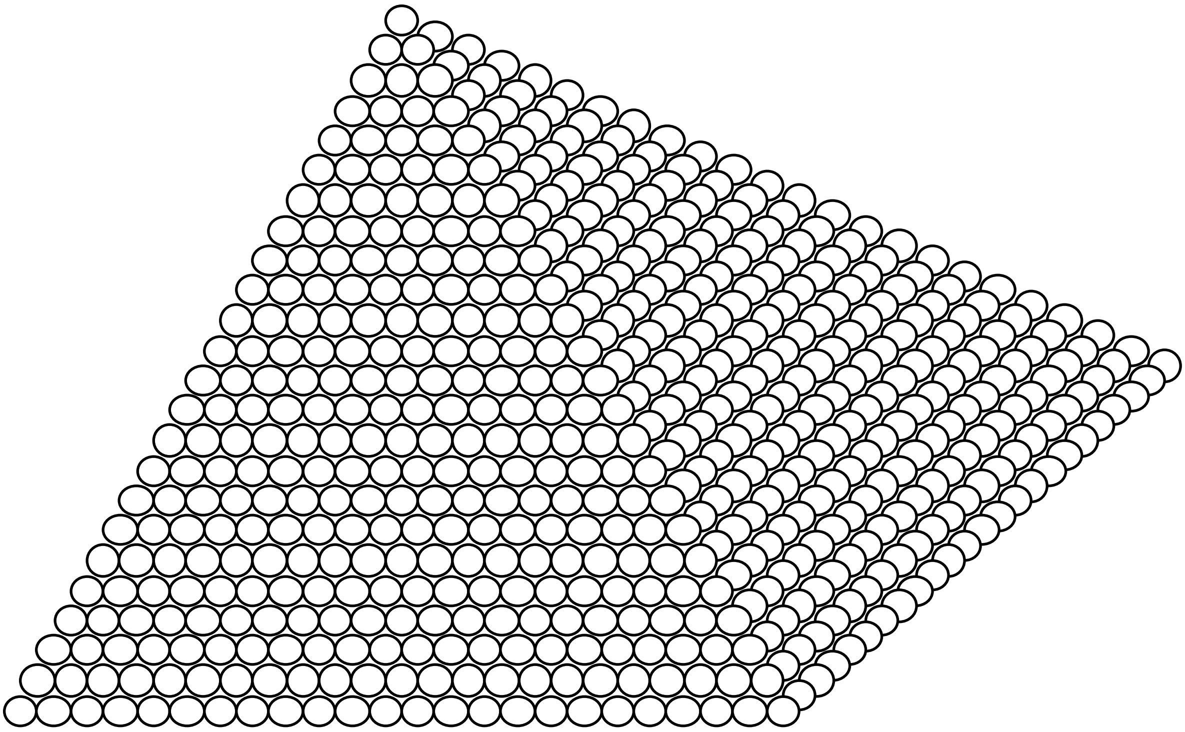
Well, I've done a few calculations and I've found that the n^{th} pyramidal number is

$$P_n = n(n+1)(2n+1)/6.$$

And I've just noticed that if you work out P_{24}, you get 24_25_49/6 = 4900, which is a perfect square. In fact when n = 24 each of n/6, n+1, and 2n+1 are squares, which is a very curious coincidence. At any rate,

$$P_{24} = 1 + 2^2 + 3^2 + ... + 24^2 = 4900 = 70^2.$$

So we can break open the 70_70 packs one at a time, and reassemble them into pyramids (**Fig.3**)."



A pyramid of base 24 can be reassembled tof orm a square of side 70. This is the *only* case, other than 1, where a pyramidal number is square

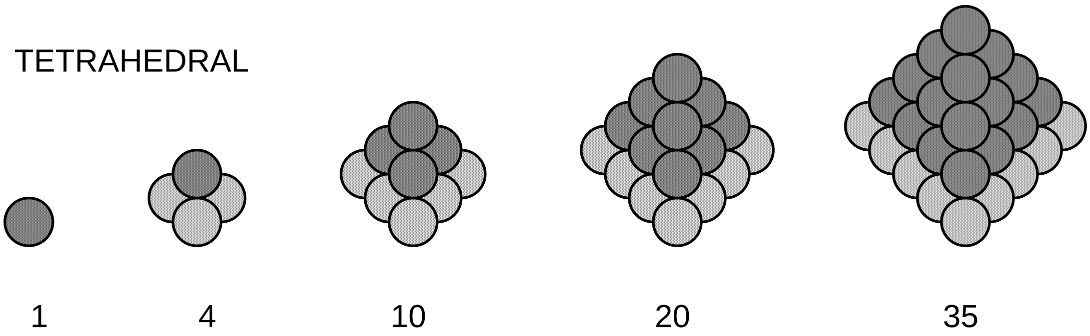
"Can you do that for any other square packs?"

"Well, the 1_1 pack becomes a pyramid of height 1..."

"That's silly. Any *other* squares?"

"Can't find any," said Billy Jo. (In fact, G.N.Watson proved in 19 that P_1 and P_{24} are the only pyramidal numbers that are perfect squares. This had been conjectured for a long time, but the proof was elusive.)

Of course, the warehouse would soon run out of 70_70 packs too, so they had to think of some other options. "What about triangular pyramids?" said Billy Jo. "Made by piling up triangles. Call them *tetrahedral numbers*, T_n. So (**Fig.4**)



Tetrahedral numbers, obtained by stacking consecutive traingular numbers.

$T_1 = 1$

$T_2 = 1 + 3 = 4$

$T_3 = 1 + 3 + 6 = 10$

$T_4 = 1 + 3 + 6 + 10 = 20$

$T_5 = 1 + 3 + 6 + 10 + 15 = 35$

and so on."

"Is there a formula?"

"Yes, $T_n = n(n+1)(n+2)/6$. The first two tetrahedral numbers are obviously perfect squares, but I can find at least one more." *Can you? As a hint, it occurs before n = 50. See ANSWERS at the end.*

"We need some other ideas," said Jego, "Or we'll run out again."

"Such as?" asked Osborne.

"Such as four-dimensional packaging."

"What's that?"

"As my old friend Albert Einstein remarked, *time* provides a fourth dimension."

"You *knew* Albert Einstein?"

"Oh yes."

"Wow."

"Albert Nugent Einstein, that is. Owns a bar in Dijon. Always keeps quoting his namesake, that funny old chap with the fuzzy hair."

"Oh.

"At any rate, suppose we provide customers with a 7-day offer. On day 1 they get a pack containing the first tetrahedral number T_1. On day 2 they get a pack containing the first tetrahedral number T_1. And so on, until on day 7 they get a pack containing the seventh tetrahedral number T_7. Then that's a four-dimensional 'tetrahedron'. *And* regular repeat sales are good for business."

"Right," said Billy Jo. "We could call it the seventh *four-dimensional tetrahedral number*

$$T^4_7 = T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7$$
$$= 1 + 4 + 10 + 20 + 35 + 56 + 84$$
$$= 210.$$

There's a formula here, too:

$$T^4_n = n(n+1)(n+2)(n+3)/24."$$

"Are any of these four-dimensional tetrahedral numbers perfect squares?" asked Osborne.

"Can't find any, except the first of course," said Jego. "Maybe we should try five-dimensional tetrahedral numbers —"

"Which we'll write as

$$T^5_n = T^4_1 + T^4_2 + T^4_3 + ... + T^4_n,"$$

said Billy Jo. "The formula is

$$T^5_n = n(n+1)(n+2)(n+3)(n+4)/120.$$

But how do you deliver a 5-dimensional package?"

"You could deliver bigger and bigger 4-dimensional packs every month," said Jego. "Each would arrive on several consecutive days, of course, being four-dimensional."

"Kind of using two different timescales, yeah."

"Right," said Osborne dubiously. He foresaw scheduling problems here.

"Are any of *those* things perfect squares?" asked Jego.

Billy Jo got out her calculator and started work. Twenty minutes later, she said "Bother."

"What's wrong?"

"Well, $T^5_{120} = 225150024$."

"So?"

"If it was one bigger it would be 15005^2. I suppose," she mused, "we could always chuck one ball away."

"No, the numbers are too big anyway. Even *we* don't stock flatpacks that big," said Osborne.

They tried 6-, 7-, and 8-dimensional tetrahedral numbers, with formulas

$$T^6_n = n(n+1)(n+2)(n+3)(n+4)(n+5)/720$$
$$T^7_n = n(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)/5040$$

$$T^8_n = n(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)(n+7)/40320.$$

You can see how the pattern goes: the numbers 720, 5040, and 40320 are factorials:

720 = 1_2_3_4_5_6

5040 = 1_2_3_4_5_6_7

40320 = 1_2_3_4_5_6_7_8

and a similar pattern holds in lower dimensions. *Among them they found one instance of a perfect square (other than 1). What was it?*

A thought struck Osborne. "We've been looking for perfect squares, but what about triangular numbers? We can break up triangular packs too. Are any of your multidimensional tetrahedral numbers also triangular?"

Billy Jo's claculator was out in a flash. "Well, there's a lot of three-dimensional ones," she said. "I can find

$T_3 = 10 = Î_4$

$T_8 = 120 = Î_{15}$

$T_{20} = 1540 = Î_{55}$

and maybe there are some more." *In fact there is just one more triangular T_n with n _ 1000. Can you find it? Are there any larger ones?*

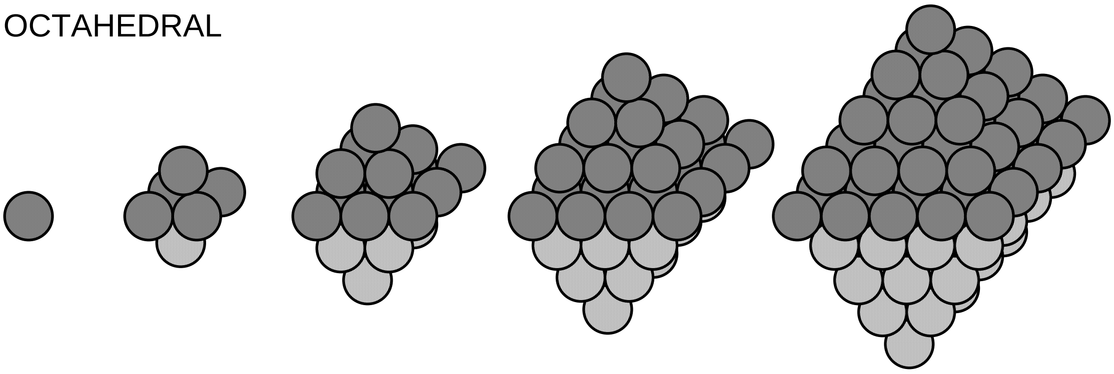
"There are also quite a few triangular T^4_n's," said Billy Jo. "In fact, I keep finding triangular T^d_n's for bigger dimensions d — there are some when d = 5, 6, 7, 8, and 9." *Can you find any of these?*

"Now that we're thinking about breaking up the triangular packs," said Jego, "we may as well go back to the original idea of square pyramidal packs. Speaking run-a-flag-up-the-pole-and-see-if-anyone-saluteswise, of course." *Are any pyramidal numbers triangular? Which?*

"I've got another idea," said Billy Jo. "How about making octahedra by building two square pyramids on the same base? One going up, one down? So we get *octahedral numbers* (**Fig.5**) which we could write as

$O_n = P_n + P_{n-1}$."

OCTAHEDRAL



1 5 14 30 55

Octahedral numbers, obtained by fitting two pyramids together, one slightly smaller than the other.

What is the formula for O_n? Are any octahedral numbers triangular? Square?

The meeting continued happily for hours, turning up innumerable strange number patterns. Eventually Jego called them to order. "I believe we have made substantial progress on agenda item 1, and we can move on to item 2."

"Which is?" asked Billy Jo.

"Changing the colour of the company's logo," said Jego. "Blue and red are terribly passé. I suggest dayglo violet with salmon pink. Unless of course anyone wants to follow next season's fashion colours and express a preference for beige and mustard..."

Billy Jo groaned.

ANSWERS

- *Square tetrahedral numbers*: $T_{48} = 19600 = 140^2$. This is the *only* square tetrahedral number other than $T_1 = 1^2$ and $T_2 = 2^2$, as proved in by

- *Square high-dimensional tetrahedral numbers*: $P^7_2 = 36 = 6^2$.

- *Triangular tetrahedral numbers:* $T_{34} = 7140 =$ Î$_{119}$. I have no idea if there are any bigger ones!

- *Triangular T^d_n's:* Here are some. There may be others.

  T^4_3 = 15 = Î$_5$
  T^4_7 = 210 = Î$_{20}$

  T^5_2 = 6 = Î$_3$
  T^5_3 = 21 = Î$_6$
  T^5_{11} = 3003 = Î$_{77}$
  T^5_{15} = 11628 = Î$_{152}$

  T^6_3 = 28 = Î$_7$
  T^6_5 = 210 = Î$_{20}$
  T^6_9 = 3003 = Î$_{77}$

  T^7_3 = 36 = Î$_8$ = ¶$_6$
  T^7_4 = 120 = Î$_{15}$

  T^8_3 = 45 = Î$_9$
  T^8_7 = 3003 = Î$_{77}$
  T^8_{10} = 24310 = Î$_{220}$

  T^9_2 = 10 = Î$_4$
  T^9_9 = 24310 = Î$_{220}$

Note the repeated appearance of particular numbers, such as 3003 and 24310. It is not entirely clear why some numbers are favoured in this manner, but see 'Mille et une coïncidences' in Further Reading.

- *Triangular pyramidal numbers:*

  Here are some. There may, for all I know, be more.

  P_5 = 55 = Î$_{10}$
  P_6 = 91 = Î$_{13}$
  P_{85} = 208355 = Î$_{645}$

- *Octahedral numbers*

  The formula is found by working out

$$O_n = P_n + P_{n-1}$$
$$= n(n+1)(2n+1)/6 + (n-1)n(2n-1)/6$$
$$= n[(n+1)(2n+1)+(n-1)(2n-1)]/6$$
$$= n[4n^2+2]/6$$
$$= n(2n^2+1)/3.$$

Coincidences include

$O_2 = 6 =$ Î$_3$

$O_7 = 231 =$ Î$_{21}$

$O_{12} = 1156 =$ ¶$_{34}$

FURTHER READING

David O. Wells, *The Penguin Dictionary of Curious and Interesting Numbers*, Harmondsworth, Middlesex 1992.

Ian Stewart, Mille et une coïncidences, *Pour La Science* **183** (January 1993) 92-95.

Ian Stewart, Un paquet à problème, *Pour La Science* **189** (July 1993) 101-103.

[This is the French translation of:

A bundling fool beats the wrap, *Scientific American* **268** #6, June 1993, 190-111.]