

The Sausage Conjecture

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It was late afternoon in the warehouse of the Spherical Sportsware Shipping Shop. Silas Golding, newly hired by SSSS, was coming to the end of his first day's work as a tennis-ball packer.

"Billy Jo, could you pass me another seven cans?" he said. "And while you're at it, how about some shrink-wrap film?"

"Just let me wrap these basketballs," said Billy Jo Rotweiler, "and then it's yours." She pushed the roll of plastic along the bench.

Silas stacked the seven cans in a line — tall cylinders each holding half a dozen tennis balls — and carefully wound the transparent plastic film around them, like a wide belt (**Fig.1a**). Then he pushed a button and watched the entire pack pass through the shrinking machine, so that the film fitted snugly — hugging the cans round the corners, and passing straight across from one to the next along the edge. But when he tried to pick the bundle up to drop it into one of the huge cardboard packing cases, it collapsed, spilling cans in all directions.

"Golding!" yelled the supervisor.

"Sir!"

"That's the hundredth time today you've done that. And you've only been working here since this morning."

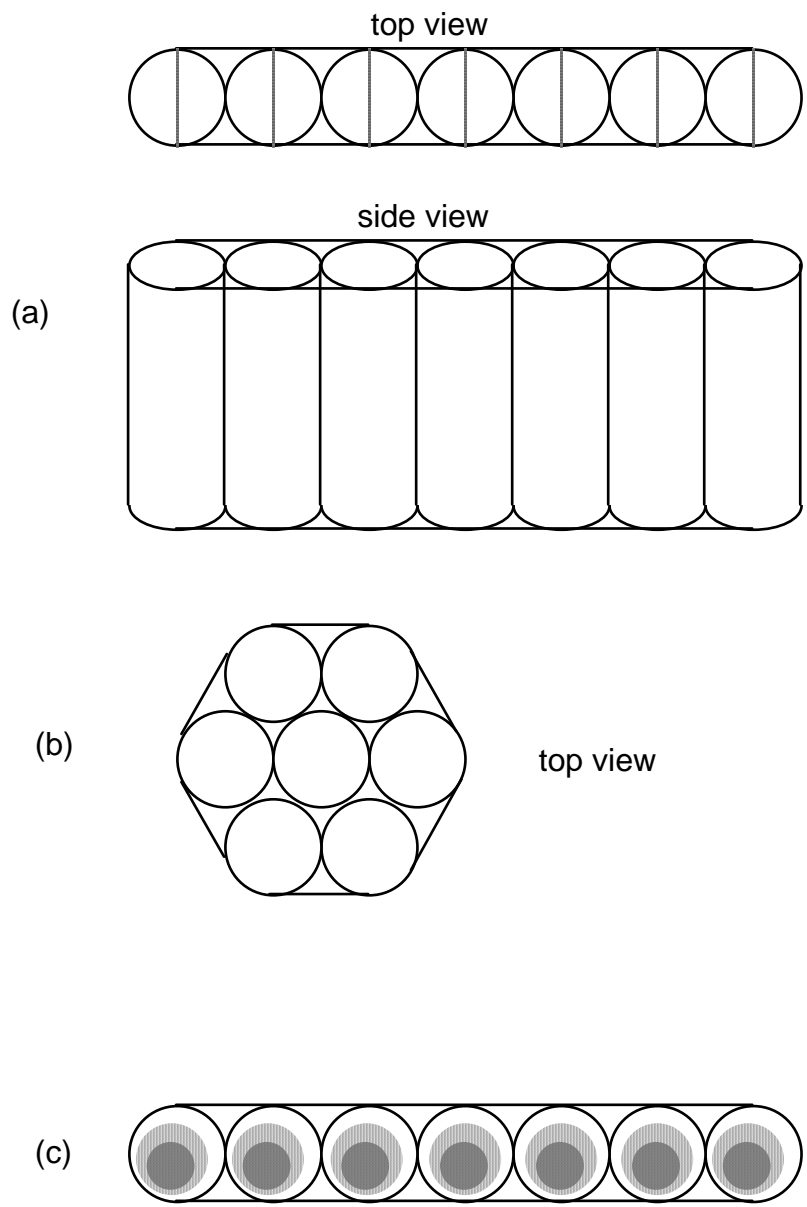
"Sorry, sir."

"Do you want to work here *tomorrow*, Golding?"

"Sir, yes sir."

"Then find a way to stop the packs collapsing. Why not try tying them in a more compact bundle?" (See **Fig.1b**). "That way the film will stay tight when you pick them up."

"But Billy Jo packs *her* basketballs in a line, sir — like a string of sausages," protested Silas. (See **Fig.1c**.)



(a) Seven cylinders arranged in a line and shrink-wrapped. (b) The same seven cylinders arranged in a hexagon and shrink-wrapped. (c) Seven spheres arranged in a line and shrink-wrapped. What are the volumes inside the wrapping?

"And she stretches the wrapping tightly just like I do, so that there aren't any dents or anything."

"I do that to keep the total volume of the pack as small as I can," said Billy Jo. "I suppose that if I could suck out some air, and get the wrapping to curve inwards, then I could make the volume even smaller — but it always pulls tight so that the pack ends up convex in shape. So I do the best I can. At any rate, *my* packs don't collapse."

"OK. Well, I thought I'd do the same with the cylindrical cans."

"But somehow, Golding, it doesn't work so well. Why?"

"Well, sir, I can see that if the total volume of the pack — including any air spaces — is as small as possible, then it won't collapse. Because if it did, you could shrink it even further, and you can't, so to speak. So we've got to think about packs whose volume — including everything inside the wrapper, air and all — is as small as possible, subject to being convex."

"I can see that the *surface area* of the bundle is a lot longer when the cans are arranged in a line," said the supervisor. "Of course that's determined by how *long* the strip of wrapping film must be. But I'm not so sure about the *volume*."

"Well," said Billy Jo, "suppose the cans have unit radius, to keep the calculation simple. If you look down on the top then effectively you're just packing a lot of circles together and wrapping an elastic band round them. The volume of your pack is proportional to the area inside the band; and the surface area of your wrapping is proportional to the length of the band. Because you don't cover the ends when you wrap the cans up."

"Right."

"So we can think about the circles, and find the area inside the band and the length of the band. It's a lot easier to think in two dimensions. Now, if you put seven circles in a line and wrap a band round to get a tight fit with convex edges, then the total perimeter is $2_+24 = 30.283$."

"Why?"

"Each straight side has length 12 — six diameters of the circles — and then there are two semicircles, each of length $_$."

"Right."

"But if you group them with one circle at the centre and the other six arranged tightly round it in a hexagon, then the perimeter is $12+2_ = 18.283$. It's a similar argument: there are six straight sides whose lengths are equal to a diameter, and six arcs, each making up one sixth of a circle."

"Yes, I can see that, Rotweiler."

"So the perimeter for the hexagonal arrangement is only a bit more than half the length of the sausage. That's why your packs keep collapsing: if you move the cans a little bit the wrapping goes loose."

"Those are the perimeters, Rotweiler. What about the volumes?"

"Should be able to use the same kind of calculation, sir... If you arrange them all in a sausage, then the total area inside the bundle can be split up as 6 $\frac{1}{2} \times 2 \times 2$ squares plus two unit radius semicircles; so that's $24 + \pi = 27.141$. Whereas if you arrange them in a hexagonal group..." her voice trailed off.

"Yeah?"

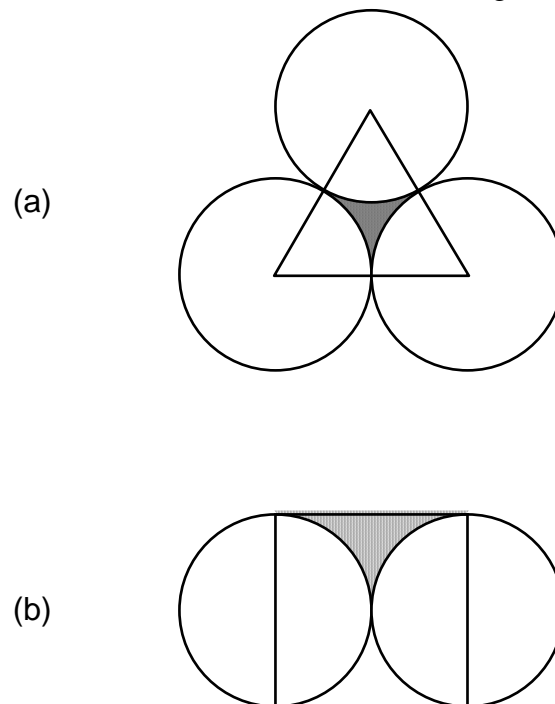
"It's kind of tricky to work it out. There are lots of funny bits with curved edges."

"Funny bits. Curved edges. Rotweiler, can't you *ever* say anything precise?"

"Isn't there a formula for things like that, sir?"

"Not in any textbooks I know of."

"Then I'll have to work them out." She grabbed a pencil and some scrap paper and began sketching. "Hmmm. Seems to me I've got to work out the areas of two kinds of shape — apart from circles. There are 'centre holes' with three curved sides (**Fig.2a**), and 'edge holes' with two curved sides and one straight (**Fig.2b**).



Finding the areas of (a) central holes and (b) edge holes. A central hole is an equilateral triangle minus three sectors of a circle that together form a semicircle. Two edge holes plus two semicircles make a square.

Ah! Now I see how to work the areas out. A central hole is really an equilateral triangle minus three 60° sectors of a circle. The equilateral triangle has sides of length 2, so its perpendicular height is $\sqrt{3}$, so its area is half the base times the

perpendicular height, which is $\frac{1}{2}\sqrt{3} = \frac{\sqrt{3}}{2}$. The three sectors together form a semicircle, of area $\frac{\pi}{2}$. So a central hole has area $\frac{\sqrt{3}}{2} - \frac{\pi}{2} = 0.161$.

"Similarly, two edge holes plus two semicircles make a 2×2 square with area 4; so one edge hole plus one semicircle has area half that, namely 2. So an edge hole has area $2 - \frac{\pi}{2} = 0.429$.

"Finally, a hexagonal bundle of seven circles has area made up from seven circles, six central holes, and six edge holes. That comes to

$$7\frac{\pi}{2} + 6(\frac{\sqrt{3}}{2} - \frac{\pi}{2}) + 6(2 - \frac{\pi}{2}) = \frac{7\pi}{2} + 6\frac{\sqrt{3}}{2} + 12 = 25.533.$$

Which is *smaller* than 27.141. See? That's why Silas's sausages collapse. He should make hexagons instead."

"Fine," said Silas. "But I don't see why it doesn't work the same way with spheres. Surely if you pack your basketballs into a tight group, then the total volume, wrapping and all, will be smaller than if they're arranged in a line."

"Yeah, but it's more complicated. You see, it's a genuinely three-dimensional problem now. I don't see how to arrange seven spheres in a tight group without there being an awful lot of wasted space inside the wrapper. But it's not so easy to work out the math. I can see that for my arrangement, in a line, the area of the wrapper is $28\sqrt{3}$ and the volume inside it is $40\sqrt{3}$ (*readers may care to work out why*). But it's harder to work them out for other arrangements. Still, I repeat: *my* packs don't collapse. So that's experimental evidence I'm right."

"This is all very well," said the supervisor, "but you often have to tie bundles with a lot more than seven cans in them. What shape will *those* have to be to minimize the volume inside the packaging, Golding?"

"Sort of — like the hexagon, sir, but bigger," said Silas.

"Like. The. Hexagon. Sir. But. Bigger," said the supervisor, plonking each word down separately. "Not exactly high precision, is it, Golding?"

"Well, no, sir..."

"And as for you, Rotweiler, you can't actually calculate any volumes at all, apart from the sausage shape that you *claim* has the smallest volume."

She shook her head apologetically.

"I want you both to spend the rest of the day in the company library, digging out any useful information you can find. I'm sure somebody must have studied the problem before. No point in reinventing the wheel, is there?"

"Yes, sir," said Billy Jo and Silas together. "We mean, no, sir." Well, it would make a change from shrink-wrapping balls.

Next morning, the supervisor appeared. "Find any *precise* results, you two?"

"Yes, sir. Quite a lot, actually, sir."

"Well then, out with them."

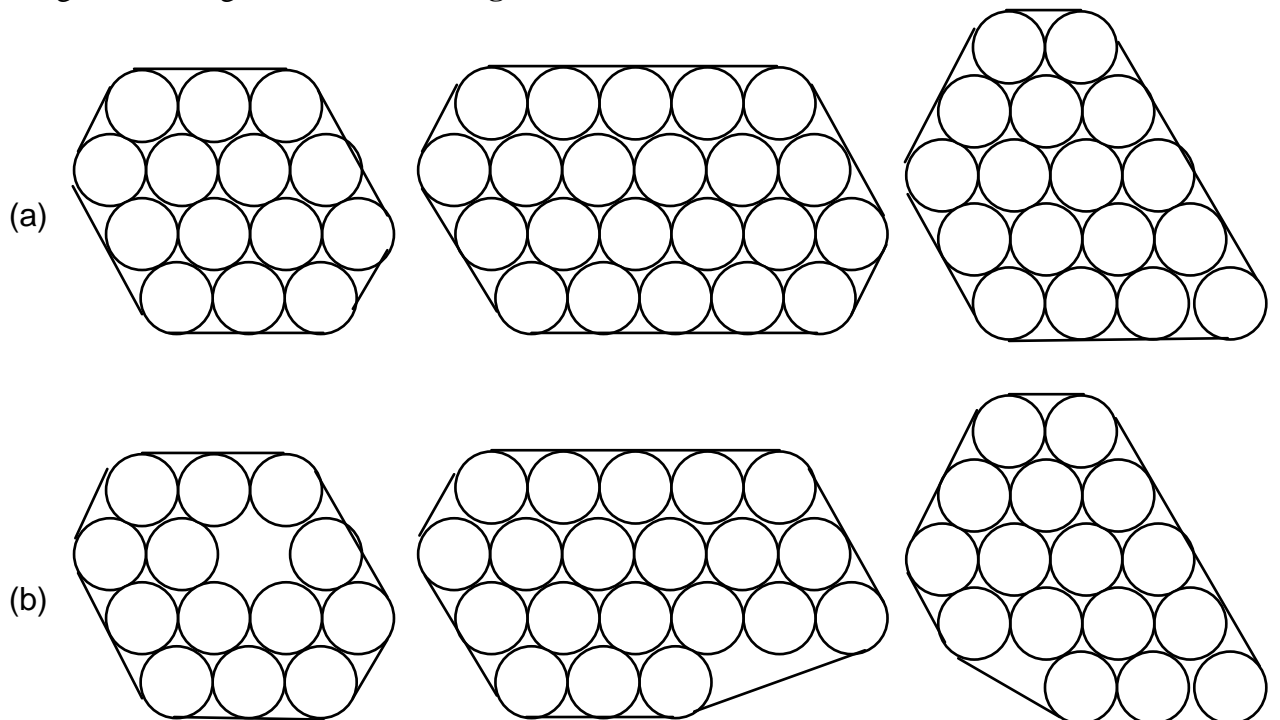
"Sir, the basic problem is to find arrangements of n -dimensional spheres in n -dimensional space that minimize the n -dimensional 'volume' of their convex hull. Er, the convex hull is the smallest convex set that contains all the spheres. In two dimensions we're packing circles in the plane, and trying to minimize the area of the convex hull; that's like packing Silas's cylindrical cans. In three dimensions we're packing spheres, like my basketballs, and trying to minimize the volume."

"Yes. Go on."

"In two dimensions the best arrangements are known to be 'as hexagonal as possible'. Well, usually."

"I thought you said you'd found some *precise* information, Rotweiler."

"Um, well, sir, it's a bit tricky because a few final details haven't been settled. I have to explain about *Groemer packings*. They're honeycomb-like packings that form hexagons — not necessarily regular hexagons, but the sides have to be parallel in pairs. Oh, and some 'sides' may not occur at all — I mean, you could have pentagons and triangles and things like that." See **Fig.3**.



(a) Some Groemer packings. (b) Some non-Groemer packings.

"They're named after Helmut Groemer of Oregon State University, who proved a fundamental theorem about them in 1960. It relates the area inside the convex hull of a circle -packing to the number of circles inside it and its perimeter. Namely, for n circles and perimeter p , the total area of the convex hull is at least

$$2\sqrt{3}(n-1) + p(1 - \sqrt{3}/2) + \sqrt{3}.$$

Moreover, the area is equal to this expression if and only if the packing is a Groemer packing."

"And?"

"Starting from this estimate, Gerd Wegner of Dortmund University pretty much solved the whole problem in 1986. He showed that the convex hull of a set of n equal circles is minimal when they are arranged in a Groemer packing whose sides are as close to being equal as possible — at least, provided n is at most 120. The theorem is also true if n is a 'hexagonal number' of the form $3k^2 + 3k + 1$, when there is a Groemer packing forming a perfect regular hexagon."

"But there are still some unsolved cases?"

"Yes, but then the gap between the best *known* results and the best that possibly could be achieved is very small."

the supervisor shook his head. "Not as precise as I had hoped, Rotweiler."

"No, sir," said Golding, leaping to her assistance. "But we never have to wrap more than 120 cans in one pack."

"True, but irrelevant to the theoretical problem," said the supervisor.

"The strange thing is that in three dimensions, the answer is quite different — at least, for a smallish number of balls."

"A Smallish. Number." said the supervisor.

"Anything up to and including 56, sir," said Billy Jo hastily. "For that many spheres, the arrangement whose convex hull has smallest volume is a *sausage* — all the balls in a straight line. (Fig.4).



If you have to to shrink-wrap this many basketballs to obtain the smallest volume pack, this is how to do it.

After that, though, the minimal arrangements get much more compact and rotund."

"Hmmm. *Rotund...*"

"Less intuitive still is what happens in spaces of four or more dimensions," said Rotweiler quickly.

"I don't think we make four-dimensional sports gear," said the supervisor.

"Perhaps when we move over to virtual reality games?" suggested Golding.

"In any case, sir, as you yourself just said, it's the theoretical point that matters."

"True. But what do you mean by four-dimensional space?"

"Well, two-dimensional space can be defined using two numbers as coordinates, (x,y) ; and three-dimensional space can be defined using three numbers (x,y,z) . So obviously four-dimensional space is just the set of quadruples of numbers (w,x,y,z) ; and n -dimensional space is the set of n -tuples (x_1,\dots,x_n) . That's it, really."

"And a *sphere* in n dimensions?"

"That's the set of points that lie within unit distance of a chosen point, the centre. There's a simple formula for the distance, a bit like Pythagoras. And of course you have to define the n -dimensional analogue of volume, but that's not too hard."

"I see."

"Yes, anyway, for four dimensions, the arrangement of 4-dimensional spheres whose convex hull has least 4-dimensional volume is a sausage for any number of balls up to 50,000. It's *not* a sausage for 100,000 balls. So the best packing involves very long thin strings of balls until you get an awful lot of them."

"Indeed, Rotweiler. An awful lot."

"Somewhere between 50,000 and 100,00, sir. Nobody knows the precise value at which sausages cease to be the best. But the really fascinating change comes at five dimensions. At least, that's the conjecture. You might imagine that in five dimensions sausages are best for, say, up to 50 billion balls, but then something more rotund has a convex hull of smaller volume; and for six dimensions the same sort of thing holds up to 29 squillion balls, and so on. But in 1975 Laszlo Fejes Tóth was led to formulate what is now called the *Sausage Conjecture*. That states that for dimensions five or more, the arrangement of balls whose convex hull has minimal volume is *always* a sausage, however large the number of balls may be."

"Good heavens," said the supervisor, genuinely surprised.

"What's so special about five dimensions?" asked Silas.

"Well, do bear in mind it's only a conjecture," said Billy Jo. "But the heuristic evidence is quite strong. The basic idea is that as the number of space dimensions increases, it gets harder and harder to fill space efficiently with spheres. The gaps are multidimensional and get quite big. If you make the wrapping poke out along too many dimensions of the space, you leave a lot of gaps, and that makes the volume quite large. Whereas if you confine the spheres to a single direction, in a sausage, then you cut down the wasted volume so much that it all works. Tóth estimated the dimension at which the numbers for this argument work out, and the break-point seems to be dimension five."

"It could all be rubbish, of course — nobody knows for certain. But it sure would be interesting to find out."

The supervisor tried hard not to look impressed. "Very well, you two," he said. "Good work. But now — back to your benches, you've got a lot of packs to wrap. Oh, and Rotweiler?"

"Sir?"

"Drive down to the grocery store and buy me a pack of sausages."

"Why, sir?"

"I want to work out why butchers always sell them in tight bundles, rather than stretched out in a straight line."

FURTHER READING

Hallard T. Croft, Kenneth J. Falconer, and Richard K. Guy, *Unsolved Problems in Geometry*, Springer-Verlag, New York 1991.

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