A Short Trek to Infinity

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Stardate 25.12.2291. The Starship Overambitious *is in transit to the Lesser Magellanic Cloud, having narrowly escaped being eaten by a Cosmic Caterpillar...* Captain Jonah T. Kink wiped his brow in relief. It had been a near thing until the immense beast had suddenly turned into a Cosmic Chrysalis. He made a mental note to find out what the lifetime of a Cosmic Butterfly was, in case it proved necessary to return to the region in future. He looked at the view screen, licked his Starfleet issue stylus thoughtfully, and added: *We have encountered a strange, glowing region of space.*

"We have encountered a strange, glowing region of space, Captain," reported Mr. Pox, his second-in-command, whose characteristic button nose and shaggy ears revealed his origin as the planet Vulgaria. Indeed his father was Vulgarian; but his mother hailed from Wapping, so Pox was only half-Vulgar.

"I've never seen such a thing before," mused Kink.

"Neither have I, Captain."

"Ha! Stumped you for once!" cried Kink in triumph.

"However," Pox continued, "it is evident that we have encountered a NIF."

"Dr. McCool probably forgot to wash his socks again."

"No, Captain; a NIF is a Newtonian Ideality Field. According to ancient legends — which I confess I have never credited until this moment — there exists in this region of space an incredibly intelligent alien race, the Weelers, who can impose upon space-time whatever structure they wish. Here the Weelers have chosen the structure of Newtonian, rather than relativistic, mechanics — an infinite, three-dimensional universe in which both space and time are absolute."

"Exactly what I was thinking," said Kink. "Ummm — Ideality Field?"

"Within the Field, idealizations such as rigid rods and point masses take on a genuine physical existence, Captain."

They watched. At the centre of the screen something almost too tiny to see was growing, expanding, faster and faster —

WHAM!

Kink picked himself up and brushed ineffectually at his uniform. "It's passed us, Pox. Lieutenant Yahoota?"

"Sir?"

"Track the anomaly and determine its current position."

"Sir." The Lieutenant twiddled some knobs on her instrument panel, becoming increasingly agitated.

"What's wrong, Yahoota?"

"The anomaly is... no longer in this universe, Captain."

"You mean it's vanished?"

"Not exactly, Sir. It... expanded to infinity. It got there exactly 17.23 seconds ago, and thereafter ceased to exist."

"Remarkable," said Pox. "The legend of the Weelers is true. Science unknown to Man or Vulgar."

"The anomaly consisted of a number of point masses," said Yahoota.

"Snotty: analysis?" Mr. Snott, the Chief Engineer, studied his instruments for a moment. "*Newtonian* point masses, ah'd hazard a wee guess... *verra* tiny particles, sor," he said, in a thick Scots accent. "Either Newtrons or Newtrinos, ah'm no sure which."

"Any evidence of unusual forces?"

"No, sor. Only the usual inverse square law gravitational attraction."

"Then the anomaly blew away to infinity in a finite period of time, unaided by any technology save for the NIF," said Yahoota.

"But — that's impossible," said Kink. "Even in a mathematical idealization, a system of gravitating point masses can't just whizz off to infinity! It would violate the law of conservation of energy."

"That is unclear, Captain, since any gain of kinetic energy might be compensated by a loss of potential energy." Kink looked perplexed. "The masses can go faster provided they are in a weaker gravitational field, which is in fact the case provided they spread further apart. Yahoota: did all the masses depart towards infinity in the same direction?"

"No, Mr. Pox, the system effectively exploded, travelling in several different directions."

"Well, in any case, they still can't do it," Kink persisted. "It would be totally illogical. You can't *get* to infinity."

Pox perked up at the sound of one of his favourite words. "Logically speaking, Captain, that is correct." Kink looked smug. "However, it can be used as a form of words to describe motion that progresses beyond any sphere of fixed radius after an appropriate period of time. In that sense, a particle *can* get to infinity. Think of a single particle travelling in a straight line at a constant speed. After infinite time, it will have removed itself from the interior of any finite sphere, effectively falling off the edge of the universe. Except, of course," he added hastily, "that an infinite universe doesn't actually *have* an edge to fall off. I apologise for that intrusion from the Wapping half of my ancestry, Captain."

"Naturally I *meant*, Mr. Pox, that the masses cannot escape to infinity in a *finite* period of time."

Pox waggled his ears in a paroxysm of Vulgar contemplation. "Possibly you are right, Captain. But a mechanism of sorts does suggest itself to me. There is no *logical* contradiction in motion that escapes to infinity within a finite interval of time."

Mr. Snott snorted loudly. "Kindly enlighten us, Mr. Pox."

"If a particle accelerates sufficiently rapidly, then it can travel an infinite distance in a finite time. Imagine that for the first second it moves at a speed of 1 kilometre per second (kps). In the first second it travels one kilometre. Now accelerate it so that for the next half second it moves at 2kps, travelling a further kilometre. Then continue in the same way, halving the time interval and doubling the speed, like this." With his stylus Pox wrote a table of figures:

Time	**Speed**	**Distance**	**Total distance**
0-1	1	1	1
1-1ï	2	1	2
1ï-1$\frac{3}{4}$	4	1	3
1$\frac{3}{4}$ - 1$\frac{7}{8}$	8	1	4
1$\frac{7}{8}$-1$\frac{15}{16}$	16	1	5
1$\frac{15}{16}$-1$\frac{31}{32}$	32	1	6

"After two seconds, the particle will have travelled an infinite distance. Its speed increases geometrically — that is, by a fixed multiple — in intervals of time that decrease geometrically. Provided the speed increases by a factor greater than that by which the interval of time decreases, the effect will be the same. I call this condition the *Geometric Growth Criterion*. The time required to reach infinity depends on the growth rates, but it will always be finite. For example, if the time halves at each stage, but the speed doubles, or trebles — but doesn't increase by, say, 50%, which is too small — then the body escapes to infinity in a finite period of time."

"But surely, Mr. Pox, that means going faster than light?"

"Yes, Snotty, but it's a Newtonian Ideality Field, not an Einsteinian one," said Kink. He drummed his fingers on the table. "Snotty, can we defend the *Overambitious* against Newtonian point masses moving at superluminal velocities?"

"Newtron bombs would do the trick, sor. But our supplies o' those are verra limited. It all depends upon the answer to a simple question. *How many bodies were there?*"

"Yahoota?"

"Not sure, sir. Not very many."

"Humph. The entire facilities of a Starfleet Cruiser and my Lieutenant says 'not very many'. *How* not very many? One? Two?"

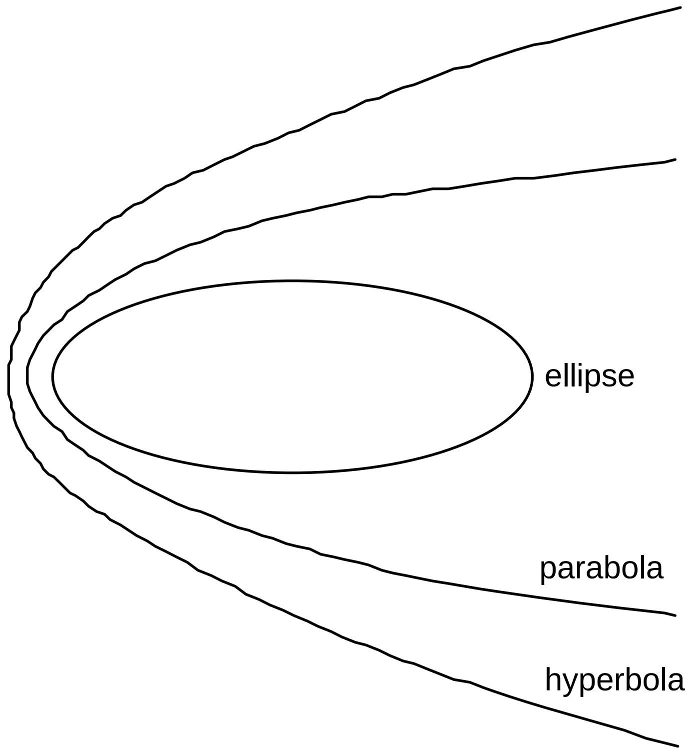
"The events happened so fast, they exceeded instrumentation limits, sir."

"It cannot be just one body, Captain," said the Vulgar. "Under Newtonian gravity, a single body, not influenced by any other forces, moves at uniform velocity in a straight line."

"It's a simple consequence of the law of conservation of momentum, sor," said Mr. Snott. "In any system of particles the vector sum of the masses multiplied by their velocities is constant. With only one particle, of constant mass, the velocity must also be a constant *vector* — uniform motion in a straight line."

"Scratch one body, Yahoota. Two, maybe?"

"No, Captain," said Pox. "With two particles, the orbits would be conic sections — ellipses, hyperbolas, or parabolas (**Fig.1**).



The three types of conic section: all possible non-collision orbits for a gravitating two-body system.

Planetary orbits are ellipses; those of comets coming in from deep space are parabolic or hyperbolic. Planets in elliptical orbits can't go off to infinity at all — they remain bound to their suns. Comets can, but the time they take to get to infinity is infinite. Instead of obeying my geometric growth criterion, comets *slow down* as they get further away."

"Three? Four? More?"

"We can rest assured, Captain, that the Weelers would use the smallest possible number of masses," said Pox. "Their extreme efficiency is legendary."

"I'll call up the information on the ship's computer, Captain." Yahoota twiddled more knobs. "Interesting... It appears that the problem was first posed in the late nineteenth century, stardate 1895 to be precise, by the mathematician Paul Painlevé —"

"No need to read it to us. Put Delphi on voice, Yahoota."

"Confirmation, sir?" The computer's synthetic voice sounded rather like a duck inside a tin can: it was generally referred to as 'Daffy' rather than the official name that Kink had used. But this was an emergency, and Kink confirmed the order. The computer's irritating voice quacked at them:

"A system has a *singularity* at some instant of time if solutions of its dynamical equations cannot be continued past that time. The simplest singularity for a system of gravitating point masses is a collision, when two masses coincide. Are there any other singularities?"

"You tell us," said Kink.

"For one or two bodies I've already explained that the answer is 'no', Captain," Pox murmured.

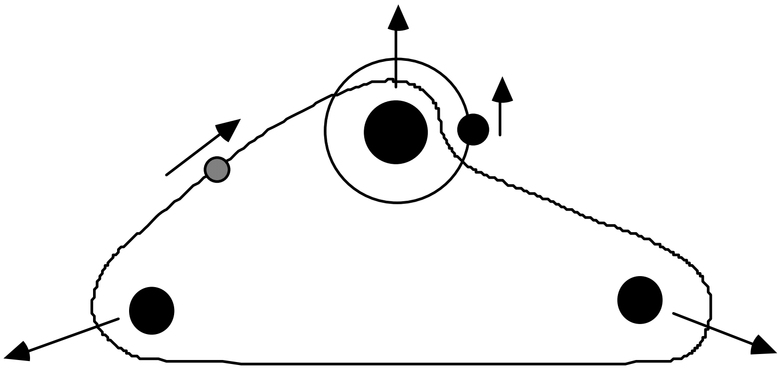
"Daffy, Pox, not you. Yahoota, ask it about three or more bodies."

"Painlevé proved that the same is true for three bodies; but he could not extend his result to four or more bodies, and he conjectured that a non-collision singularity, or *pseudocollision*, might be possible. The simplest pseudocollision occurs when one or more of the bodies escape to infinity in a finite period of time. Another possibility is that one or more of the bodies starts to oscillate ever more wildly as time approaches some particular value. In the twentieth century the combined researches of H.Von Zeipel, Richard McGehee, Donald Saari, and H.J.Sperling proved that in order to have a pseudocollision in the many-body problem, *both* types of bad behaviour must occur simultaneously. That is, some bodies must move off to infinity in finite time, *and* oscillate wildly."

"Whizzes and wobbles," muttered Kink. "Sorry," he added, when Daffy politely waited for further commands. "Continue."

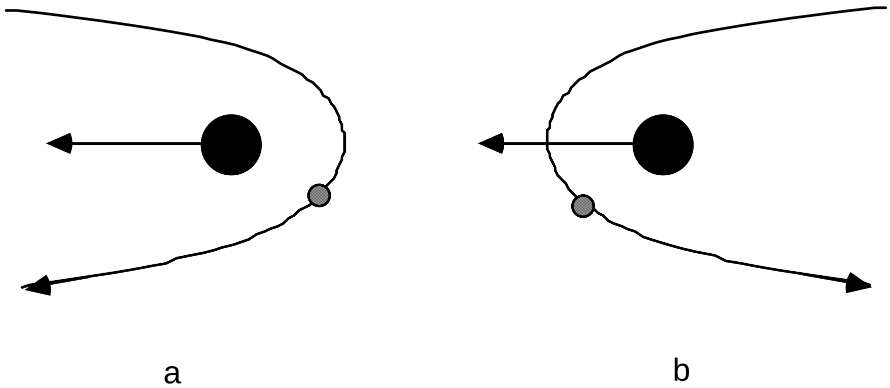
"Saari also proved that pseudocollisions in the four-body problem are infinitely rare. If you choose initial conditions at random, then the chance of a pseudocollision is zero. John Mather and McGehee did discover a pseudocollision in a system of four bodies confined to a line — but only after an infinite number of collisions, assumed to be followed by elastic bounces. Then, in Stardate 1984, Joseph Gerver came up with a scenario that might produce an escape to infinity, requiring the active participation of *five* bodies..."

Although the bodies are point masses, let's use astronomical imagery to remind ourselves of their relative masses. Take three stars, one more massive than the others, and arrange them in a triangular constellation with an obtuse angle at the heaviest star. Set the stars moving outwards, away from the centre of the triangle. Add a tiny asteroid that orbits round the outside of all three stars, approaching them very closely (**Fig.2**).



Joseph Gerver's scenario for the Great Escape.

As the asteroid passes the most massive star, arrange for it to undergo a 'slingshot' effect, gaining energy from the star, and decreasing that star's energy by the same amount (**Fig.3a**). It can then, on subsequent encounters with the other two stars, transfer energy to them by a reverse slingshot (**Fig.3b**).

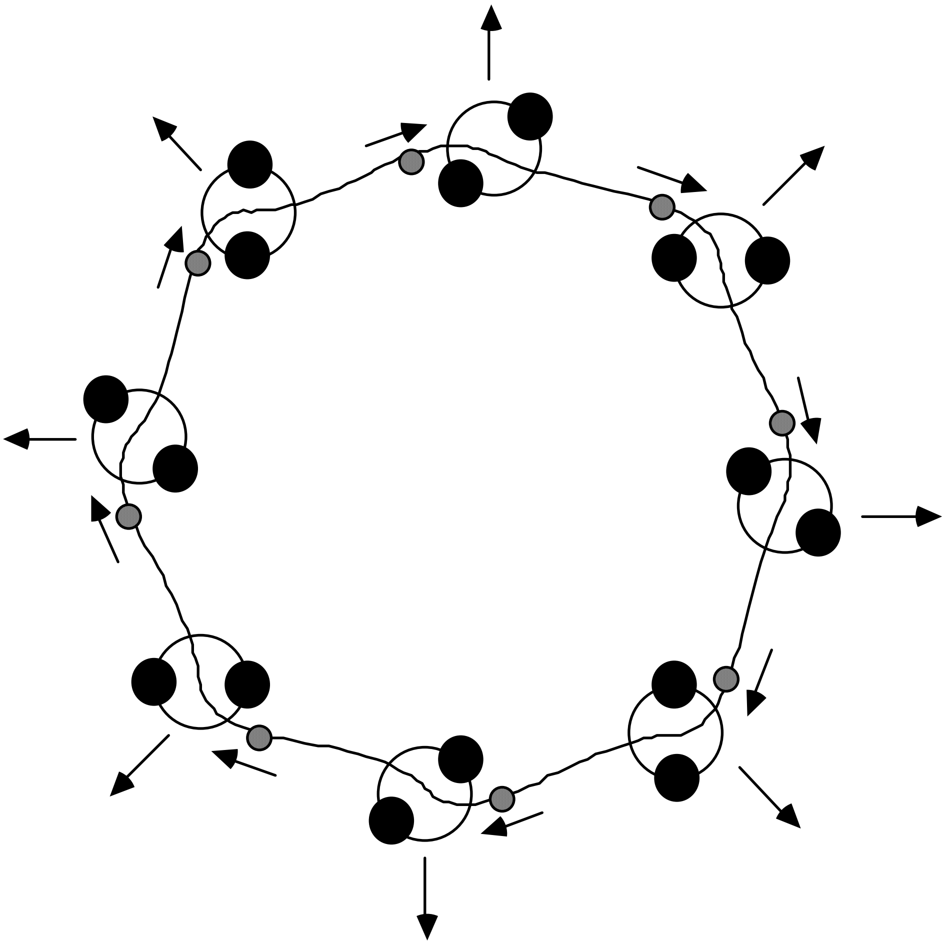


The slingshot effect. (**a**) How the asteroid gains energy at the expense of the star. (**b**) How the star gains energy at the expense of the asteroid.

As a result, the speeds of the asteroid and the two less massive stars can be increased. If only it were possible to speed up the heaviest star as well, then the triangle could be made to expand according to Pox's geometric growth criterion, and the entire system would escape to infinity after a finite period of time.

Unfortunately, the law of conservation of energy prevents the heaviest star from speeding up too. But Gerver found a legal loophole. Add a *fifth* body, a planet that orbits the most massive star. Instead of making the star lose energy, make the *planet* lose energy — so much energy that the star can *gain* some. That is, arrange the slingshot so that the asteroid whips past both planet and star, slowing the planet down but speeding up the star. Now, on each circuit of the asteroid, the stars and asteroid speed up, and the planet slows down, moving closer to its star. The energies balance, but the triangle grows — and it grows geometrically fast, so that all three stars do indeed escape to infinity in a finite period of time, taking the asteroid and the planet with them!

Gerver observed that although this scenario is plausible, the calculations needed to prove that it really works become so messy that the proof cannot be brought to a conclusion. That situation persisted until 1989, when he used an idea suggested by Scott Brown to prove that for large enough *n* the 3*n*-body problem really does permit an escape to infinity. The configuration employed is a more symmetric version of the triangle of stars, and consists of *n* pairs of binary stars, all having the same mass. The orbits in each pair are nearly circular, and the centres of mass of the pairs lie at the vertices of a regular *n*-gon. A further *n* planets, all having the same mass (but much smaller than the mass of the stars) move approximately along the edges of the polygon (**Fig.4**).



A symmetric configuration of 3*n* bodies (illustrated here for *n* = 8) provides a more tractable mathematical problem. When *n* is sufficiently large, it can be proved that there exist initial positions and velocities for which all bodies disappear to infinity in finite time.

Each time a planet approaches a binary star, it gains kinetic energy via the slingshot effect; the binary star compensates by losing kinetic energy and moving to a tighter orbit.

The planet also transfers momentum to the binary star, causing it to move outwards, away from the centre of the polygon. Because of symmetry, all *n* binary stars are affected in exactly the same manner at exactly the same time. At each stage,

- the polygon grows;
- the planets move faster;
- the binary stars close up into tighter orbits.

By suitably adjusting the number of bodies, their masses, and their initial positions and velocities, it is possible to set up a system in which

- infinitely many slingshot events occur, separated by times that decrease geometrically
- the polygon grows geometrically at each slingshot, at a faster rate than the times decrease

Infinitely many slingshots occur during a finite time, by the end of which the entire system has escaped to infinity.

"The role of symmetry is interesting, Captain," said Pox, having asked Daffy to print out a complete, logical proof. "Effectively, it reduces the problem from 3*n* bodies to 3. Once you have determined the positions and velocities of one binary star and one planet, the symmetry determines those of the remaining 3*n*-3 bodies. In other words, the problem reduces to one about three disconnected 'bodies' — each a regular *n*-gon of point masses — moving under an extremely complicated set of forces. However, it still looks rather intractable to me."

"It becomes tractable for large enough *n*, because the forces simplify in that case," said the computer.

"How large *n*?" asked Kink.

"Gerver did not determine the precise value."

"Confound it!" yelled the Captain of the Starfleet Cruiser *Overambitious*. "Can't *anyone* give me a simple answer to a simple question?"

"In Stardate 1988, Z. Xia also proved that the 5-body problem has a solution in which all five bodies escape to infinity in finite time. His scenario is different from Gerver's, but it also involves symmetry."

"At last," said Kink. "So five point masses are enough."

"And four almost certainly aren't," said Pox. "Since the Weelers always employ the optimal means to achieve their ends, the anomaly must have contained precisely five bodies."

"Snotty, do we have five Newtron bombs?"

"I'll just go and check, sor. It depends whether any have been dismantled. The platinum gauze nose cones make great coffee filters, you see, sor."

"Brilliant," said Kink. But it was never easy, being a Starfleet Captain. Certainly not on *his* ship. "The important question now becomes, what *are* the Weelers' ends? Why have they set up a Newtonian Ideality Field?"

"Captain!" Yahoota looked worriedly over her shoulder, while her fingers tapped the keyboard. "The NIF is running wild and has begun to interact with the *Overambitious*'s drive! We are already moving at warp speed 1 and it's growing — geometrically!"

"Over geometrically decreasing intervals of time?" asked Pox eagerly, drooping his ears in self-congratulation when Yahoota nodded.

"Warp 2!...4!...8!...16!..."

Kink turned to his Chief Engineer. "Snotty, get the drive shut down immediately, before we all disappear to infini—

FURTHER READING

Joseph L Gerver, A possible model for a singularity without collisions in the five body problem, *Journal of Differential Equations* **52** (1984) 76-90.

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