The Never-Ending Chess Game

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What is mathematics? It seems impossible to give a succinct definition. The best one I know is 'what mathematicians do'. Similarly, the best definition of 'mathematician' is 'someone who does mathematics'. There are some people who think that unless you can define something, then it doesn't really exist — or, at the very least, you're not entitled to talk about it. Some of these people are mathematicians. The dilemma is not peculiar to mathematics. For example, if you try to define 'business', you're likely to end up with 'what businessmen do'; and then a businessman is 'somebody who does business'.

I was mulling the question over, and it occurred to me that it's possible to say a bit more. A businessman doesn't just 'do business'. What makes a person a businessman is that they see *opportunities* for doing business where most other people would not. They have a businesslike worldview. In the same way, a mathematician is somebody who sees opportunities for doing mathematics where most other people don't. They have a mathematical worldview.

Here's a case in point. Anyone who plays chess knows that some games just peter out into pointless situations where neither player looks like winning, nothing constructive can be done, and there's no obvious way to end the game except to agree a draw. But what if the other player won't agree? Then the game might go on indefinitely. The bodies who frame the laws of chess have foreseen such situations, and many different rules have been proposed to force games to end. The classic law is 'the game shall be drawn if the Player prove that 50 moves have been made on each side without checkmate having been given and without any men having been captured or Pawn moved'.

However some recent computer analyses have shown that there are some end-games where one player can force a win, but this involves making more than fifty moves without capturing any pieces or moving pawns, so the laws of chess are forced to specify certain exceptional situations. Any law that specifies a limit on the number of moves permitted under particular conditions runs exactly the same risk, so it would be nice to find a different approach altogether. One proposal, made some time ago, was that the game should end if the same sequence of moves, in exactly the same positions, is repeated three times in a row. (Do not confuse this with the standard law that if the same *position* occurs three times, the player facing it can claim a draw. But note that this law does not oblige them to do so.) It might be a short sequence, or a long one: the proposed rule was careful not to specify the length. You can make out a good case that any violation of this three-in-a-row rule ought to end the game. The question is: are there pointless games that do *not* violate it? It's here that the mathematical worldview sees an interesting question. Can a game of chess go on forever, without checkmate, and without repeating the same sequence of moves three times in a row? (A game that goes on forever is certainly a pointless one.)

Chess is kind of complicated, so any mathematician worth their salt would try to simplify. Suppose we focus on just two possible moves, represented by the binary symbols 0 and 1. Can a sequence of 0's and 1's go on forever without any finite block repeating three times in a row?

It turns out that there are many ways to produce such a sequence, which I'll call a *tripleless sequence*. The first was invented by Marston Morse and ?? Hedlund while investigating a problem in dynamics. Begin with a single 0. Follow it by the complementary sequence (every 0 changed to a 1 and vice versa), which here is just 1, so you get 01. Then follow that by its complementary sequence, and so on, building up an infinite sequence like this

0

0**1**

01**10**

0110**1001**

01101001**10010110**

and continuing the process indefinitely. I've written the compementary sequences in boldface for clarity.

This sequence is genuinely tripleless, but a proof of that property is tricky. There is a more explicit tripleless sequence for which the proof is a bit easier. In order to describe it, we need some terminology. Recall that an even number is a multiple of 2, whereas on odd number is one greater than a multiple of 2; more simply, even numbers are of the form 2m and odd numbers of the form 2m+1. We need a similar terminology for multiples of *three*. Say that a number is

treble if it is a multiple of 3 (that is, of the form 3m)

soprano if it is one higher than a multiple of 3 (that is, of the form 3m+1)

bass if it is one lower than a multiple of 3 (that is, of the form 3m-1).

Every whole number is either treble, soprano, or bass. If a number is soprano (equal to 3m+1 for some m) then say that m is its *precursor*. For example 16 = 3x5+1 is soprano, and its precusor is 5, which is bass.

Using this terminology, we can write down a recipe for a sequence that never repeats a block three times in a row:

- The first term is 0.
- The nth term in the sequence is 0 if n is treble.
- The nth term in the sequence is 1 if n is bass.
- If n is soprano, with precursor m, then the nth term in the sequence is equal to the mth term.

The first three rules tell us that the sequence goes

010*10*10*10*10...

where the pattern *10 repeats indefinitely, and the starred entries aren't yet determined. The fourth rule lets us work upwards along the starred entries. For example entry 4 is the same as its precursor, which is entry 1, and that's a zero. Entry 7 is the same as its precursor, which is entry 2, and that's a one; and so on. Because the precursors are smaller, their values will have been worked out already, so rule 4 does indeed determine all the stars.

These rules lead to what I'll call the *choral sequence*

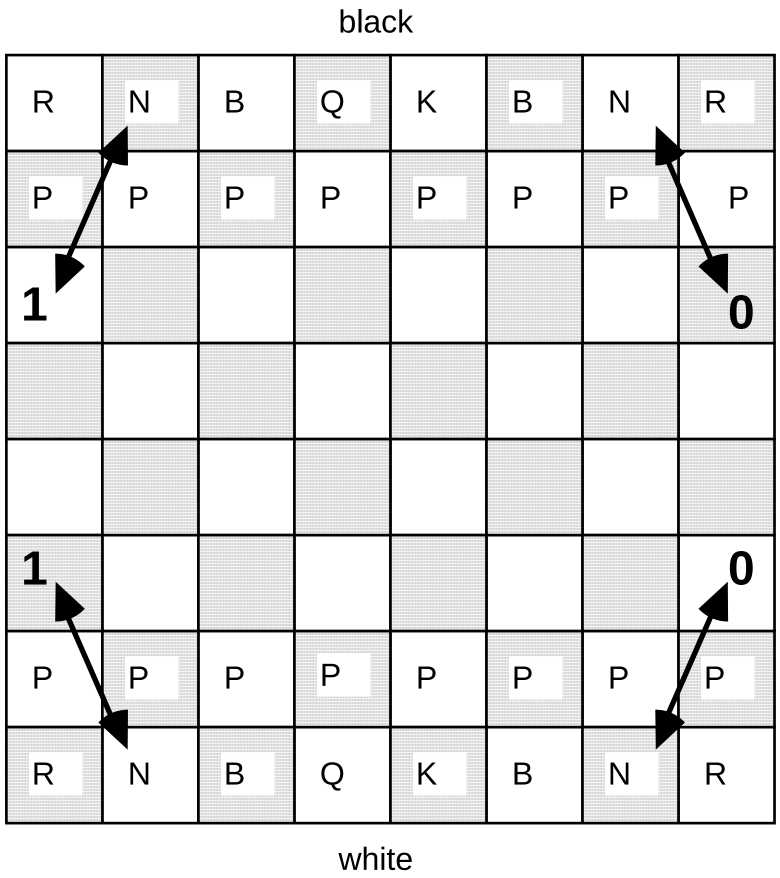
010 **0**10 **1**10 **0**10 **0**10 **1**10 **0**10 **1**10 **1**10 **0**10 **0**10 **1**10 ...

I've grouped the terms in threes to show the structure more clearly, and put the soprano terms in boldface. The choral sequence has the curious property that the boldface terms reproduce the entire sequence exactly.

There are lots of double repetitions of blocks in the choral sequence: for example it starts 010 010, and the first 18 terms repeat the sequence 010010110 twice. But no block ever repeats *three* times (see BOX 1), so it is tripleless.

How does this help with the chess problem? There are many more moves in chess than just two; and if you pick two (say advancing the king's pawn and moving the king's rook three spaces forwards) it's not at all clear that the sequence corresponds to *legal* moves. The way to get round this is actually quite simple; but you might like to think about it before reading on.

OK, here goes. Suppose that both players confine themselves to moving one or other of their knights out and back as in Fig.1.



In the never-ending chess game, only knights ever move, back and forth between two squares. Symbols 0 and 1 show the corresponding terms in the tripleless ' choral sequence'.

Depending on their current position, either the outward move or the backward move is available for each knight. Suppose the players use the sequence of 0's and 1's to determine their moves, with '0' interpreted as 'move the king's knight' (KN) and '1' as 'move the queen's knight' (QN), like this:

0 White moves KN (out)
1 Black moves QN (out)
0 White moves KN (back)
0 Black moves KN (out)
1 White moves QN (out)
0 Black moves KN (back)

and so on.

It's not exactly an exciting chess game, but it's a perfectly legal one — in the sense that each individual move is legal. And because of its relation to the choral sequence, it is clear that it goes on forever without ever repeating the same sequence of moves three times in a row. In fact, more strongly, it doesn't repeat the same sequence of *pieces* (KN or QN) three times in a row. So if you're looking for a truly watertight chess law to terminate pointless games — one that is proof even against players colluding to play stupidly but legally — that old proposal doesn't work.

This particular problem motivates mathematicians to ask related questions about symbol sequences. For example, is there a sequence of 0's and 1's that never repeats a block *twice* in a row? Does the answer change if you're allowed more symbols, say 0,1,2? Recreational mathematicians can have fun turning such questions into analogous ones about chess; for example, can a legal game of chess continue indefinitely without any block of moves repeating twice in a row?

Its effect on the framing of chess laws is unlikely to be great, however, because chess players normally have a sensible objective in mind and we don't (yet?) know how to capture that condition mathematically. Which brings us back to where we began. To define something is to draw very tight boundaries around it. My own view is that everything really interesting has fuzzy edges, which just get fuzzier when you try to pin them down with a formal definition. Indeed it is this fuzziness of boundaries that provides lawyers with a living: even such appparently black-and-white concepts as 'dead' or 'female' turn out to have fuzzy boundaries. Despite this, every chess player knows what a 'sensible' game of chess is — even though they can't *define* 'sensible'. Similarly every businessman knows what business is; and every mathematician knows what mathematics is.

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BOX 1   Proof that no block occurs three times in a row.

Call successive symbols 0 or 1 the terms of the sequence, and say that the nth term is treble, bass, or soprano if n is.

1. No block of length 1 is repeated three times, because any three consecutive terms must include both a treble term and a bass term, which are different.

2. No block of length 2 is repeated three times, because any six consecutive terms contain a block of the form 0*1, but neither 010101 nor 101010 does.

3. If a block of length 3 is repeated three times, then it contains three soprano terms whose precursor terms are all the same and consecutive — which is ruled out by step 1.

4. If a block whose length is a multiple of 3 — say 3k — is repeated three times, then a similar argument shows that a block of length k must have been repeated three times earlier in the sequence.

5. The only remaining case is when a block whose length is a least 4, and not a multiple of 3, is repeated three times. In this case the proof gets more complicated. To see the idea, suppose that the length is 4, so the sequence includes a block of the form abcdabcdabcd. One of the first three terms must be treble; suppose for example that it is c. Then the block actually goes ab0da**b**0d**a**b0**d**. But every third term after the first 0 — marked in bold — is also treble, so b = a = d = 0 and the entire block goes 000000000, which is ruled out by step 1. Similar arguments hold if a or b is treble. A more convoluted version of the same kind of argument works for any block whose length is not a multiple of 3.

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FURTHER READING

Paul R. Halmos, *Problems for Mathematicians Young and Old*, Dolciani Mathematical Expositions 12, The Mathematical Association of America, Washington DC, 1991.