

Unbang the Bane and the Army of the Antigods

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"So you've finally decided to show your miserable face again."

Pnerd, apprentice to Rockchopper Rocknuttersson of the Obelisker's Guild, had returned from repairing the Quagville marketplace with brand new domino-shaped paving slabs (see "Dolmens, dominos, et damiers", *Visions Mathématiques, Pour La Science* **175**, May 1992, 100-103.). "I don't pay you to take extended holidays."

"You don't pay me to take short holidays, Rocky. Come to that, you don't pay me at all."

"Guild Rules. No pay when business is bad." The yard was piled high with unused rocks, stones, slabs, gravel, slates, boulders, chippings, obelisks, and pebbles.

"So you always say. I thought the recession had ended."

"Bottomed out, Pnerd, not ended. Just bumping along waiting for a consumer-led recovery that doesn't suck in imports but instead gives a much-needed boost to our manufacturing sector."

"What manufacturing sector?"

"Good point."

"I do hear there's been a sudden boom in the manufacture of wings for pigs."

"Quite. Were the priests satisfied?"

"Rocky, since when have you seen a satisfied priest?"

"I mean, did they pay the bill?"

"After I threatened to inflict them with festering nostril plague throughout eternity and beyond, yes."

"Excellent. Surprisingly prompt. I was expecting you to have to resort to the curse of the Mad Cow at the very least."

"Yup. I think the quality of our paving discouraged them from bargaining."

"Maybe." Rocky finally noticed a large sack that Pnerd was dragging along behind him. "What's in that?" Without waiting for an answer, he yanked on the drawstring and tipped out some of the contents. "Stones? For gods' sake, Pnerd, don't you think the Obeliskers' Guild has enough stones already without you sucking in imports from Quagville? You've single-handedly wrecked the economic recovery! Well, you would have if there had actually been one," he added grudgingly.

"It's a game," said Pnerd. "It's a present for Snitchswisher Wishsnitchersdorter." He blushed. "It only cost five hundred copper obles."

"What did we get from the priests, Pnerd?"

"Um. Five hundred copper obles, Rocky. But it's worth it. I bet Snitchswisher's never come across anything remotely like this."

Rocky eyed the sack. "All that work and you've only got a sack of stones to show for it?"

"I've also got the sole marketing rights for the Antigod Game throughout the whole of Bogpool."

"The what?"

"Antigod Game. This thing in the sack."

Rocky flung a bevel-flanged chisel the length of the yard. "You're an idiot."

"Maybe. But just think about this. One set of the Antigod Game sells for five hundred copper obles — and it's made entirely from stone. Any kind of stone."

"Say that again." Pnerd did. "We are kind of overstocked on stone," said Rocky, gesturing at the heaps crammed all around him. He put a friendly arm round Pnerd's shoulder. "Tell me more about this game."

"I've got to explain it to Snitchswisher when I give it to her. Why don't you come along too?" They set off for Dead Cat Swamp.

When they arrived, Snitchswisher was digging for lump-eels, to pickle against a hard winter. Pnerd shyly proffered his gift. Snitchswisher tugged the drawstring open. "Oh, Pnerd, it's a Antigod Game!" Pnerd looked downcast. "Just what I always wanted! How clever of you to guess! You just can't get them around these parts, you know. Somebody ought to start a business making them, they'll sell like — um — pickled lump-eels."

She helped Pnerd unpack the game, which consisted of a large slab ruled into squares, and thousands of flat circular black stones. They placed the slab on the ground and began arranging the stones, one in each square.

"You've missed one," Rocky pointed out.

"No, there has to be one gap," said Pnerd.

"Why?"

"Because each move in the Antigod Game consists of jumping a stone over an adjacent stone into a gap, and removing the stone that is jumped over. Without a gap to start with, nothing can move."

"Oh."

" 'Adjacent' means 'next in a horizontal or vertical direction'. Not diagonally."

"Naturally. And the aim of the game is?"

"To get rid of as many of the stones as possible," said Pnerd.

"Actually," said Snitchswisher, "there's a lot more to it than that. According to ancient Nome legends, the Antigod Game should be played on an infinite board, starting with just one square empty, and the player is supposed to remove *all* of the stones. In an infinite sequence of moves, of course. It's an old numerosophistic conundrum. The Antigod Game is a re-enactment of the Uncreation of the Antigods by Unbang the Bane —"

"The what of the which by the who?"

"The Nomes," Snitchswisher explained, "believe that originally every nook and cranny in the present universe was crammed full of antigods, which prevented space, time, and matter from existing. Then Unbang the Bane banished the antigods beyond the outermost reaches of infinity, leaving room for the universe to exist — and so it did. He observed that when one antigod bypassed another into an empty region of ordinary space-time, then they mutually annihilated each other, releasing cosmic energy that became manifest as extra empty regions of space-time."

"A very curious discovery."

"Yes. We numerosophists call it the Standard Model of Renormalizable Banishment in Count 'em Mechanics. Of course it relies upon all antigods having the same count 'em numbers — things like mess, sin, harm, and unorthodoxy. Which is why all the stones are black and they all obey the same rules."

"Well, obviously," said Rocky.

"So, can it be done?" asked Pnerd.

"Can what be done?" said Rocky.

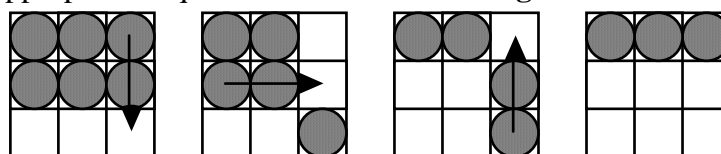
"Can all the antigods be banished, of course."

"Oh, yeah. That. Right. Snitchswisher?"

"Yes, it can. The key idea is the Rule of Triadic Annihilation."

"The what?"

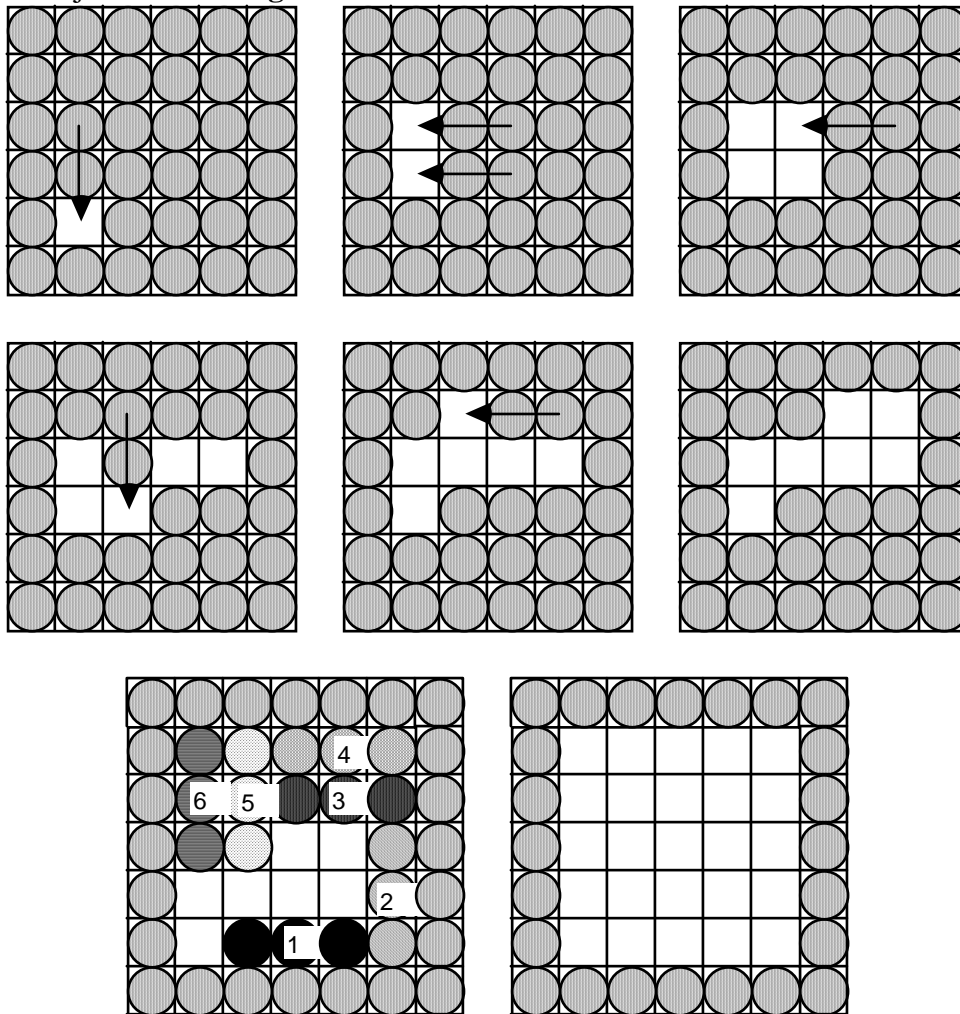
"If a line of three antigods sits next to another line of three, and if there is an empty square adjacent to the end of the line, then the first line of three antigods can be removed by an appropriate sequence of moves." See **Fig.1**.



Triad move clears out a row of three.

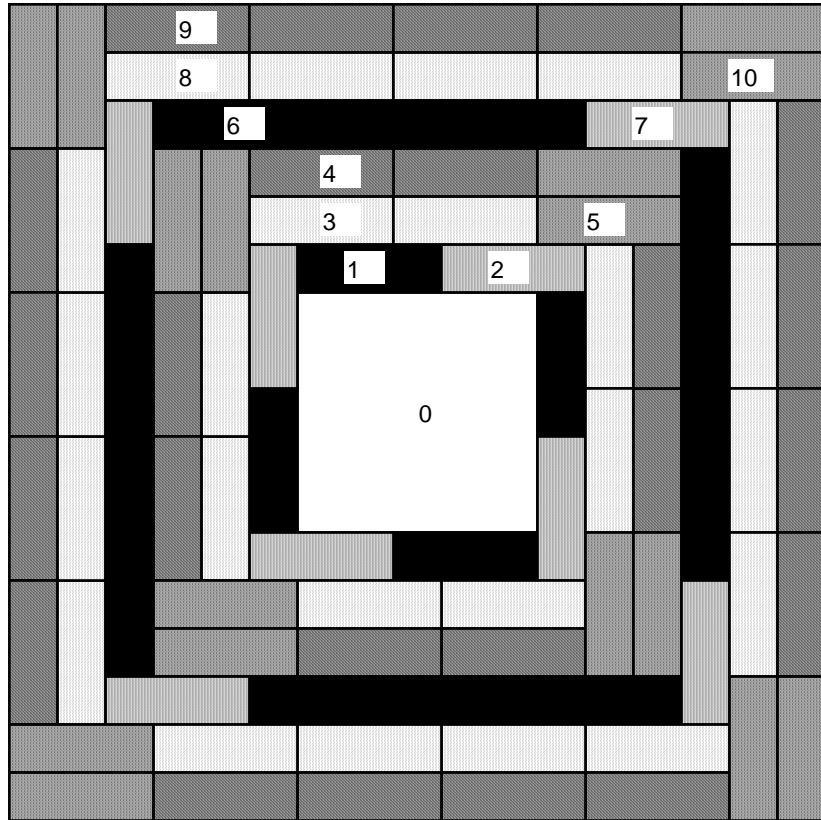
"This is very useful, because once you've created a hole in the array of antigods, you can keep stripping out sets of three using these *triad moves*.

"With that understood, the first step is to create a square hole, of size 5x5. That's done by making a few simple moves and then performing six triad moves to clear away any residual junk." See **Fig.2**.



Creating an empty 5x5 square. Six triad moves are marked, to be carried out in numerical order.

"Then you can keep enlarging the hole, in a systematic way, by making triad moves along its edges. There are several ways to do this, but one method is to enlarge the hole to sizes 7x7, 11x11, 13x13, 17x17 — increasing the size alternately by 2 and by 4." She demonstrated her method: see **Fig.3**.



How Unbang emptied the whole of the universe. Sequences of triad moves successively enlarge a square hole. Numbers indicate order of removal: all triads on the boundary of a given square that have the same colour as the numbered triad are removed before proceeding to the next number.

"Why the difference?"

"You have to follow a different pattern if the size of the hole is one greater than a multiple of three, or one less. Otherwsie the triads don't fit. The cases alternate: 5 is one less than a multiple of 3, 7 one greater, 11 one less, 13 one greater, and so on."

"Right."

"Of course," said Snitchswisher, you can't actually *make* an infinite board. So—"

"Board?"

"The grid of squares."

"That's a slab, not a board," said Rocky.

"Numerosophists call it a board—"

"By Grungegrinder's fifth armpit, I should know a slab when I see one," said Rocky, reddening. "Board's made from trees, not rocks. Flaking Obelisker's Guild member ought to know a flaking slab when he flaking well sees—"

"Well, yes, but let's call it a board just to please the lady," said Pnerd quickly. "And don't forget to put three copper obles in the cursing-box when we get back."

"Oh, yes. Sorry."

"There's a lot more to the old Nome legends," said Snitchswisher, trying to change the subject. "One group of heretics holds that Unbang the Bane was forced to engage in a cosmic struggle with Big Crunch, and only managed to clear out half the universe. We wrongly think he cleared it *all* out because the part we live in is surrounded by a lot of ordinary space-time. But out there in the distance, so far away that we cannot see it, is the Great Wall. There the normal universe ends, and a semi-infinite half-plane of antigods wait to reclaim their rightful domain."

"How far?"

"Oooh, enormous distances. Maybe a thousand mule-years."

"What's a mule-year?"

"The distance a mule can cover in a year." They fell silent, awestruck by the vastness of the cosmos. Until Pnerd said "With or without carrots?", which everyone else held to be irrelevant to the general order of magnitude.

"That's why the sky is black," said Snitchswisher. "The stars that we see are all on our side of the Great Wall. Behind them is the infinite blackness of the Wall itself."

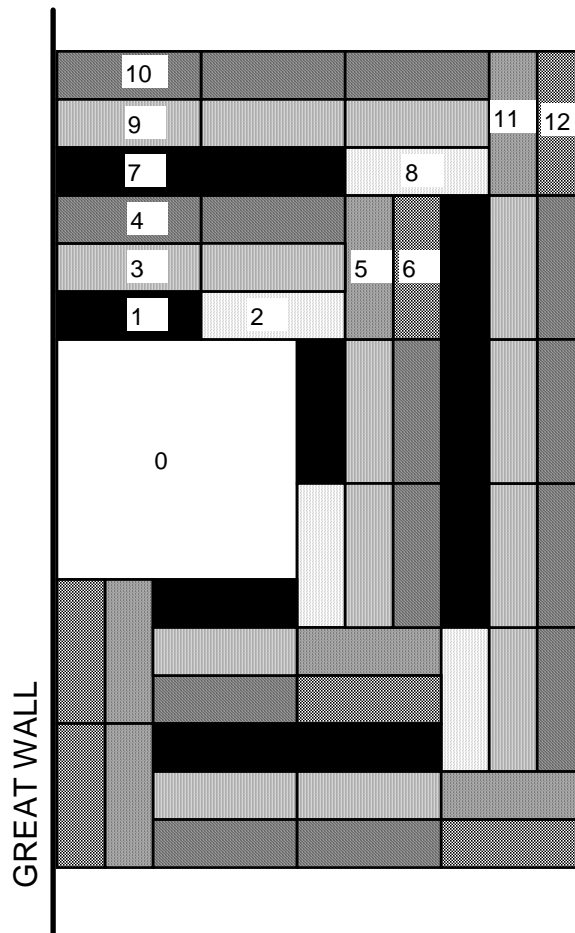
"Sounds reasonable," said Pnerd.

"Irrefutable," said Rocky. "Except you said we can't see it."

"We can't. It's black."

"Oh, yeah."

"Numerosophists have discovered how Unbang could clear out a half-plane of antigods, using much the same method as before," said Snitchswisher. "Again he clears out a 5x5 square and then enlarges it. But now he uses a different sequence of triad moves, leaving a wall of antigods on one side of the initial square." See **Fig.4**.



How Unbang emptied half the universe.

"It's very worrying that the antigods might yet destroy our universe," said Rocky.

"Not really," said Snitchswisher. "You see, there has been a numerosophistic discovery of vast cosmological significance."

"Which is?"

"That if the antigods occupy only a half-plane, they can never penetrate more than a few squares into the normal universe."

"Explain that more carefully, please," said Pnerd.

"Suppose you start with an army of antigods, all on one side of a line representing the Great Wall," said Snitchswisher. "Then Big Crunch makes a finite sequence of moves, aiming to send at least one of his antigods as far as possible into the universe on the far side of that line. The question is, how big must the army be in order to send one antigod 0, 1, 2, 3, 4, or 5 squares across the great Wall?"

"By making some sequence of moves according to the usual rules?"

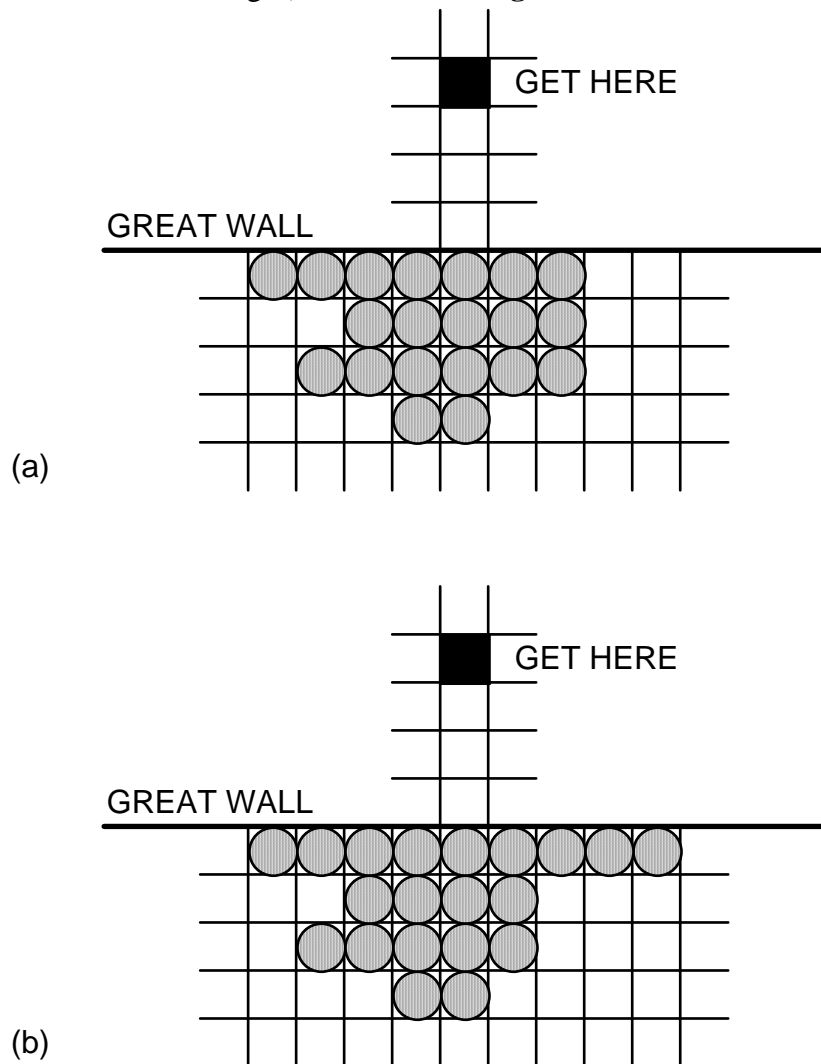
"Exactly. Now it's not hard to see that the size of army needed to send an antigod 0, 1, 2, or 3 squares across the great Wall is 1, 2, 4, and 8 respectively."

"I'll take your word for it."

"Good. The '1' for 0 squares is of course just a convention: there has to be an antigod to move, even if it can't go anywhere. At any rate, how big do you think the army need be to send an antigod 4 squares across the Great Wall?"

"Well, the obvious guess is 16."

"I agree. But the correct answer is 20. Precisely two configurations of 20 antigods (apart from mirror images) work." See **Fig.5**.



The two possible configurations for an army of 20 antigods that is capable of sending one of its members four squares out from the Great Wall.

"Now, given that the pattern of sizes goes 1, 2, 4, 8, 20, what comes next?"

"Unbang alone knows."

"No, I do too. *Infinity*. And actually, even an infinite army won't work. You can't send an antigod five squares across the Great Wall, with any finite sized army, or any infinite one, that starts completely behind the Wall."

"Um — doesn't the infinite case follow from the finite one?"

"I'll make a numerosopher of you yet, Pnerd! Why?"

"Well, because if an antigod is going to get five squares out, then it has to actually get there after a finite sequence of moves. I mean, if every finite sequence only got it four squares out — or less — then in the limit of infinitely many moves it can't get any further than the limit of a sequence of numbers of size four or less, and that limit is also four or less."

"Right... So?"

"Well, if it makes a finite sequence of moves it only *uses* a finite part of the army. So you may as well work with a finite army to begin with."

"Excellent. Got that, Rocky?"

"No. But don't let it bother you, I believe you anyway." He paused. "Except that I don't see why you are so sure that no possible army can get five squares out."

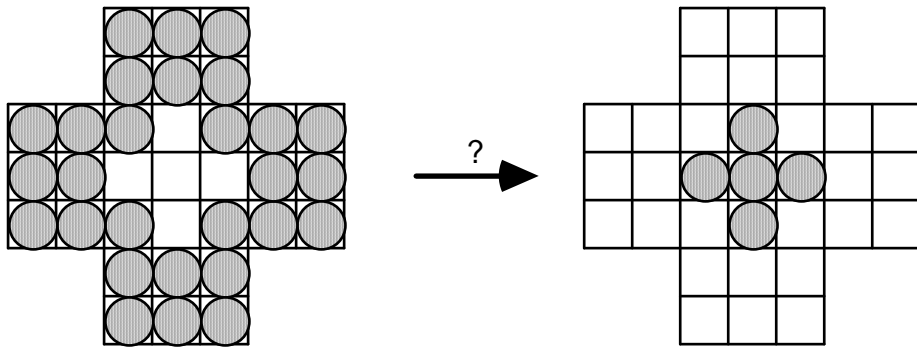
Snitchswisher smiled. "Pagoda functions," she said.

"You'd better put a copper oble in the cursing-box for that," said Rocky.

"No, it's not a curse: it's an idea for proving things about the Antigod Game. What you do is, you assign a numerical *value* to each square of the board. The value of a set of antigods is the sum of the values in the squares that they occupy. Such an assignment is called a *Pagoda function* if each valid move either leaves it the same or decreases it."

"Eh?"

"Let me give you an example. Suppose the board is a finite one, shaped like a cross. All squares are full except for a small +-shape in the centre made from five squares. Can you find a series of moves that will end up reversing the position, so that the only stones that are left occupy precisely those five squares?" See **Fig.6**.



An Antigod Game puzzle.

Pnerd and Rocky tried for ages, but couldn't solve the puzzle.

"Good, isn't it?" said Snitchswisher. "Challenge your friends, keep them busy for hours. The thing is, it's impossible. **Fig.7** shows a Pagoda function for such a board.

		-1	0	-1		
		1	1	1		
-1	1	0	1	0	1	-1
0	1	1	2	1	1	0
-1	1	0	1	0	1	-1
		1	1	1		
		-1	0	-1		

Pagoda function showing that the puzzle is impossible.

You can check that no move can increase the total value of a position. All you have to observe is that whenever three values lie in a row, like a b c, then $a+b \leq c$ and $b+c \leq a$. Because if antigod a jumps over b into a hole at c, then the value for those three squares changes from $a+b$ to c , and everything else is unchanged; and similarly if antigod c jumps over b into a hole at a.

"Given that, you can see that the total value for the initial position is 4, because each 'arm' of the cross contributes $1+1+1+(-1)+0+(-1) = 1$ and there are four arms. On

"Now, the value of any position that includes an antigod on the square with value 1, plus perhaps some other squares, so it is 1 or greater. On the other hand, the total value of all squares behind the Great Wall can be worked out algebraically by summing various geometric series. First, note that

$$f^2 + f^3 + f^4 + \dots = f^2/(1-f),$$

but by the Thaumaturgic Equality $1-f = f^2$, so this sum is equal to 1. Therefore, multiplying by f^{n-2} , we find that

$$f^n + f^{n+1} + f^{n+2} + \dots = f^{n-2}$$

for any $n \geq 2$. So the total value of the first row behind the Great Wall is

$$(f^5 + f^6 + f^7 + \dots) + (f^6 + f^7 + f^8 + \dots) = f^3 + f^4 = f^2(f+f^2) = f^2.$$

Multiplying by f , the total value of the second row is f^3 , the third row is f^4 , and so on. Therefore the total value of *all* the squares behind the Great Wall is

$$f^2 + f^3 + f^4 + \dots = 1.$$

"Finally, since each square behind the wall has a nonzero value, any finite army has total value *less* than 1. So no series of moves can increase it to a value 1 or more, which is what we must achieve if we wish to move one antigod five squares beyond the Great Wall. In short, Big Crunch can never regain more than a tiny part of what he has lost."

"Amazing," said Pnerd. "Reassuring," said Rocky. "Now, what really intrigues me is this business of an infinite slab. Seems to me we could shift an awful lot of stone if we included an infinite slab in every copy of the Antigod Game."

"Smart thinking, boss," said Pnerd.

FURTHER READING

Elwyn R. Berlekamp, John H. Conway, and Richard K. Guy, *Winning Ways* volume 2, Academic Press 1982.